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THE SYMMETRICAL FREE VIBRATIONS OF A THIN ELASTIC PLATE WITH INITIAL CONDITION AS A GENERALIZED FUNCTION

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In this paper the generalized function method is utilized in solving an initial value problem of the symmetrical free vibrations of a thin elastic plate with initial condition as a generalized function. It is substantiated that this new approach to the symmetrical free Vibrations of a thin elastic plate is very simple, rigorous and useful from the point of view of sufficient generality and applicability. The Cauchy problem has been solved by the application of Hankel transform of generalized functions. The need for generalized functions arise when the displacement and velocity distribution are caused due to sudden application of impulsive forces.

1. INTRODUCTION

The formulations of boundary value problems are generally characterized by the fact that their initial and boundary conditions are sufficiently smooth. In many practical problems there may be singularities in the initial and boundary conditions. Such problems cannot be solved with the help of conventional functions. These problems can be correctly specified with the help of generalized functions.

In this paper we shall solve a Cauchy problem with a modification in one initial condition as generalized functions by the application of Hankel transform of generalized functions.

2. PRELIMINARIES

The continuous linear functionals $\langle f, \phi \rangle$ on some fundamental space are called generalized functions (Gel'fand and Shilov 1968, p. 82). Hence in contrast to the conventional functions, generalized functions are not defined in themselves but depend on a selected space.

Let $I = (0, \infty)$ and $x \in I$. H_μ denotes the linear space consisting of all smooth complex-valued functions $\phi(x)$ on I such that for every pair of non-negative integers m and k, the numbers

$$\gamma^{\mu}_{m,k}(\phi) = \sup_{x \in I} \left| x^m . (x^{-1}D)^k \, x^{-\mu-(1/2)} . \phi(x) \right| \qquad \ldots(2.1)$$

exists. The set of semi-norms $\{\gamma^{\mu}_{m,k}\}$ generates the topology of H_μ. The dual space H'_μ of H_μ is the space of generalized functions defined on H_μ. $\mathcal{E}(I)$ denotes the space of all complex valued smooth functions defined on I. Its dual $\mathcal{E}'(I)$ is the space of distributions with compact support on I (Zemanian 1968, p. 30). The ordinary Hankel transformation h_μ is an automorphism on H_μ whenever $\mu \geqslant -\frac{1}{2}$. The generalized Hankel transformation h'_μ on H'_μ is defined by

$$\langle h'_\mu f, \phi \rangle = \langle f, h_\mu \phi \rangle \qquad \ldots(2.2)$$

where $\phi, h_\mu\phi \in H_\mu$ and $f \in H'_\mu$ (Zemanian 1968, p. 141). When $f \in \mathcal{E}'(I)$ and $\mu \geqslant -\frac{1}{2}$ the Hankel transform $F = h'_\mu f$ of f takes on the form

$$F(y) = \langle f, \sqrt{xy}, J_\mu(xy) \rangle. \qquad \ldots(2.3)$$

The Cauchy problem is the problem of finding a solution u of the system of linear partial differential equations

$$\frac{\partial u_j}{\partial t} = \sum_{k=1}^{N} p_{jk}\left(t, i\frac{\partial}{\partial x}\right) u_k + f_j \qquad \ldots(2.4)$$

$(j = 1, 2, 3, \ldots N)$ and $i = \sqrt{-1}$ under the initial conditions

$$u_j(x, 0) = u_{0j}(x) \ (j = 1, 2, \ldots, N) \qquad \ldots(2.5)$$

where $f_j(x, t)$ and $u_{0j}(x)$ are given functions and p_{jk} $(j, k = 1, 2, \ldots, N)$ is a linear differential operators of orders $\leqslant p$ (Friedman 1965, p. 165). $u = u(x, t)$ is called a generalized solution (Friedman 1965, p. 167) over a testing function space Φ of (2.4) and (2.5) if u satisfies the following conditions :

(i) For every fixed t, $0 < t \leqslant T$, $u(x, t)$ is an element of Φ';

(ii) u satisfies (2.4) in the weak sense of Φ', i.e. for every $\phi \in \Phi$, $0 < t \leqslant T$,

$$\frac{d}{dt}\langle u(x, t), \phi(x) \rangle = \operatorname*{Lt}_{h \to 0} \left\langle \frac{u(x, t+h) - u(x, t)}{h}, \phi(x) \right\rangle;$$

(iii) u satisfies (2.5) in the following sense :

$$\operatorname*{Lt}_{t \to 0} \langle u(x, t), \phi(x) \rangle = \langle u_0(x), \phi(x) \rangle,$$

for any $\phi \in \Phi$.

3. Statement of the Problem

The problem we have considered, in particular is the Cauchy problem of symmetrical free vibrations of a thin elastic plate (Sneddon 1972, p. 330) with a modification in one initial condition. The problem may be stated as follows :

Find the generalized function $w_t(p)$ defined on the domain

$$\Omega = \{(\rho, t) \mid 0 < \rho < \infty \text{ and } -\infty < t < \infty\}$$

which depends parametrically upon the time variable t and satisfies the equation

$$b^2 \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho}\right)^2 w_t(\rho) + \frac{\partial^2}{\partial t^2} w_t(\rho) = 0. \qquad \text{...(3.1)}$$

of the symmetrical free vibrations of an elastic plate of infinite radius; where $b^2 = D/2\sigma h$, D is the flexural rigidity, σ the density of the plate, $2h$ the thickness of the plate,

with the initial conditions :

(a) $w_t(\rho)$ converges to $f(\rho) \in H_0'$ as $t \to 0$,

(b) $\frac{\partial w_t}{\partial t}$ converges to $0 \in H_0'$ as $t \to 0$.

In eqn. (3.1) the differentiations with respect to ρ are generalized differentiations whereas that with respect to t, is parametric differentiation.

4. Solution of the Problem

Let $u_t(\rho) = \sqrt{\rho}\, w_t(\rho)$, then eqn. (2.1) becomes

$$b^2(M_0 N_0)^2\, u_t(\rho) + \frac{\partial^2 u_t(\rho)}{\partial t^2} = 0. \qquad \text{...(4.1)}$$

Applying Hankel transformation of order zero to (3.1) and interchanging h_0 with $\frac{\partial^2}{\partial t^2}$ we get

$$b^2 \rho^4 U_t(r) + \frac{\partial^2}{\partial t^2} U_t(r) = 0 \qquad \text{...(4.2)}$$

where $h_0 u_t(\rho) = U_t(r)$. The solution of above equation is given by

$$U_t(r) = A(r) \cos(bt\rho^2) + B(r) \sin(bt\rho^2) \qquad \text{...(4.3)}$$

where $A(r)$ and $B(r)$ are unknown generalized functions which do not depend on t. Let $h_0 f(\rho) = F(r)$. In view of boundary conditions (a) and (b) we have additional conditions :

$$U_t(r) \to h_0(r) \text{ as } t \to 0$$

and

$$\frac{\partial U_t(r)}{\partial t} \to 0 \text{ as } t \to 0.$$

Hence with respect to the above initial conditions we get $A(r) = F(r)$ and $B(r) = 0$. Then eqn. (3.3) becomes

$$U_t(r) = F(r) \cos (bt\rho^2).$$

Thus the required generalized solution is

$$U_t(\rho) = h_0^{-1}\, U_t(r). \qquad \ldots(4.4)$$

To justify this, we first show that, for each fixed t, $U_t(r) \in H_0'$ on $0 < r < \infty$, since $f(\rho) \in H_0'$, $F(r) \in H_0'$. Now to prove $U_t(r) \in H_0'$, it is sufficient to prove that for each fixed t, $\cos (bt\rho^2) \in \mathcal{O}$, where $\mathcal{O}$ is the space of multipliers (Zemanian 1968, p. 134) for H_u.

Since $(\rho^{-1}D_\rho) \cos (bt\rho^2) = O(\rho^{-\nu})$ as $\rho \to \infty$. Therefore $(\rho^{-1}D_\rho)^\nu \cos (bt\rho^2)$ is bounded on $0 < \rho < \infty$ which proves that $\cos (bt\rho^2) \in \mathcal{O}$.

Because of these results, $u_t(\rho)$ given in (3.4) is a generalized function in H_0' on $0 < \rho < \infty$ depending parametrically on t. We write it as a functional on H_0 by

$$\langle\, u_t(\rho), \phi(\rho) \,\rangle = \langle\, F(r) \cos (bt\rho^2), \Phi(r) \,\rangle \qquad \ldots(4.5)$$

where $\phi \in H_0$ and $\Phi = h_0\phi$.

Now we shall prove that $u_t(\rho)$ as defined in (3.5) satisfies the differential eqn. (3.1) in the sense of equality in H_0'. Let $\phi(\rho)$ be any function belonging to H_0 and $\Phi(r) = h_0\phi(\rho)$ then by sec. 5.5, eqn. (8) of Zemanian (1968, p. 143) we get

$$b^2 \langle\, M_0N_0u_t(\rho), \phi(\rho) \,\rangle = b^2 \langle\, \rho^4 \,.\, U_t(r), \varphi(r) \,\rangle. \qquad \ldots(4.6)$$

On the other hand by definition of the parametric differentiation $D_t = \dfrac{\partial}{\partial t}$ we see that

$$\begin{aligned}
&\langle\, D_t^2\, u_t(\rho), \phi(\rho) \,\rangle \\
&= D_t^2 \langle\, u_t(\rho), \phi(\rho) \,\rangle \\
&= D_t^2 \langle\, u_t(r), \Phi(r) \,\rangle \\
&= D_t^2 \langle\, F(r), \Phi(r) \cos (bt\rho^2) \,\rangle \\
&= \langle\, F(r), \Phi(r)\, D_t^2 \cos (bt\rho^2) \,\rangle \\
&= \langle\, F(r), \Phi(r)\, (-\, b^2\rho^4) \cos (bt\rho^2) \,\rangle \\
&= \langle\, U_t(r)\, (-\, b^2\rho^4), \Phi(r) \,\rangle \\
&= -\, b^2 \langle\, U_t(r)\, \rho^4, \Phi(r) \,\rangle
\end{aligned}$$

$$\therefore \quad \langle D_t^2 u_t(\rho), \phi(\rho) \rangle = - b^2 \langle U_t(r) \rho^4, \Phi(r) \rangle. \qquad \ldots(4.7)$$

Hence from (4.6) and (4.7) we conclude that $u_t(\rho)$ satisfies eqn. (4.1) in the stated sense.

Now we shall verify that, the generalized solution $u_t(\rho)$ satisfies the condition (a). To prove that the above condition is satisfied, we have to prove that, for any $\phi \in H_0$,

$$\langle u_t(\rho), \phi(\rho) \rangle \to \langle f(\rho), \phi(\rho) \rangle \qquad \ldots(4.8)$$

as $t \to 0$. From (4.5), we have

$$\langle u_t(\rho), \phi(\rho) \rangle = \langle F(r), \Phi(\rho) \cos (bt\rho^2) \rangle.$$

To prove (4.8), it remains to prove that $\Phi(\rho) \cos (bt\rho^2)$ converges in H_0 to $\Phi(\rho)$ as $t \to 0$ and the limit (4.8) follows directly from these results and the fact that $F(r) \in H_0'$.

The second paragraph of sec. 5.3 of (Zemanian 1968, p. 134) shows that our assertion concerning $\Phi(\rho) \cos (\rho^2 tb)$ will be established as soon as we prove that

$$\frac{\cos (b\rho^2 t)}{1 + b\rho^2} \to \frac{1}{1 + b\rho^2} \text{ as } t \to 0 \qquad \ldots(4.9)$$

and for each positive integer ν,

$$(\rho^{-1} D_\rho)^\nu . \cos (b\rho^2 t) \to 0 \qquad \ldots(4.10)$$

where in every case the convergence is uniform on $0 < \rho < \infty$.

Since $(\rho^{-1} D_\rho)^\nu \cos (b\rho^2 t) = O(\rho^{-\nu})$, the result (4.10) follows as $\rho \to \infty$. On the other hand, to show (4.9), we know that for any $\epsilon > 0$, there exists a $R < \infty$ such that for all $\rho^2 < R$ and $-\infty < t < \infty$

$$0 \leqslant \frac{1 - \cos (b\rho^2 t)}{1 + b\rho^2} < \frac{2}{1 + bR} < \epsilon.$$

Having fixed R this way, we restrict ρ to $0 < \rho^2 < R$ and t to $| t | \leqslant \pi/R$, then

$$0 \leqslant \frac{1 - \cos (\rho^2 bt)}{1 + b\rho^2} \leqslant 1 - \cos (bRt) \to 0$$

as $t \to 0$.

Therefore, there exists a $T > 0$ such that, for all $| t | < T$, we have

$$0 \leqslant \frac{1 - \cos (b\rho^2 t)}{1 + b\rho^2} < \epsilon, \; 0 < \rho < \infty.$$

Since ϵ is arbitrary, we see that

$\Phi(\rho) \cos (b\rho^2 t)$ converges in H_0 to $\Phi(\rho)$.

Hence our assertion.

Similarly $\frac{\partial u_t(x)}{\partial t} \to 0$ as $t \to 0$, in the sense of H_0'.

5. Concluding Remarks

The choice of $f(\rho) \in H_0'$ and $0 \in H_0'$ as the initial displacement distribution is quite justifiable, because for instance the displacement and velocity distribution are caused due to sudden application of impulsive force, the distributions of displacement and velocity of free surfaces at initial moment can not be correctly specified by conventional functions. We may be able to know about $f(\rho)$ and $0 \in H_0'$ from the effect they later on produce. It is such situations which are depicted by generalized functions since in our choice $f(\rho)$ and $0 \in H'$ has a compact support, it justifies the phenomena that beyond certain bounded and closed domain the initial displacement and velocity distribution is zero, because however large the magnitude of the impulsive forces may be its effect beyond certain closed and bounded domain will be zero provided that are visualising that the expanse of free surfaces is fairly large.

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