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THE ABSOLUTE CESÀRO SUMMABILITY OF LAGUERRE SERIES

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In this note a theorem on absolute Cesàro summability of Laguerre series at the frontier point is established by using Stieltjes integrals and adopting a technique different from that of Hille (1926) in his classical convergence theorem on Laguerre series.

1. DEFINITIONS AND NOTATIONS

Let $\Sigma\, a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$ and let σ_n^k denote the nth Cesàro means of order k of $\{s_n\}$. The series $\Sigma\, a_n$ is called absolutely summable (C, k), or summable $|\, C, k \,|$, $k > -1$, if

$$\sum_n |\, \sigma_n^k - \sigma_{n-1}^k \,| < \infty.$$

The Fourier-Laguerre series of a function $f(x) \in L\,[0, \infty)$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x) \qquad \text{...(1.1)}$$

where

$$\Gamma(\alpha + 1)\binom{n+\alpha}{n} a_n = \int_0^{\infty} e^{-y} y^{\alpha} L_n^{(\alpha)}(y)\, f(y)\, dy \qquad \text{...(1.2)}$$

and $L_n^{(\alpha)}(y)$ denotes the Laguerre polynomials of order $\alpha > -1$.

2. INTRODUCTION

On ordinary Cesàro summability of series (1.1), Gupta's (1973) following result (1973) is worth noting :

Theorem — For $k > \alpha + \frac{1}{2}$, $\alpha > -1$, the series (1.1) is (C, k)-summable at $x = 0$ with the sum $f(0)$, if

$$\int_0^t |\, g(u) \,|\, du = o(t^{\alpha+1}) \qquad \text{...(2.1)}$$

and

$$\int_1^{\infty} e^{u/2} u^{-(3k+1)/3} \mid g(u) \mid du < \infty \qquad \ldots(2.2)$$

where

$$g(u) = \{\Gamma(\alpha + 1)\}^{-1} [f(u) - f(0)] e^{-u} u^{\alpha}.$$

None seems to have so far tackled the problem of absolute Cesàro summability of Laguerre series. In this note using Stieltjes integrals, we attempt to establish a result on $| C, k |$-summability of the series (1.1) at the point $x = 0$ for $k > \alpha + \frac{1}{2}$, $\alpha > -1$. To achieve this end we use a technique different from that of Hille (1926) in his convergence theory of Laguerre series. We write

$$\phi(y) = f(y) - f(0).$$

3. Theorem

For $k > \alpha + \frac{1}{2}$, $\alpha > -1$, the series (1.1) is $| C, k |$-summable at $x = 0$, provided $f(x)$ is such that

$$\int_0^t | d\phi(y) | \leqslant A t^{\alpha+1}, \; 0 < t \leqslant \omega < \infty \qquad \ldots(3.1)$$

$$\int_{\omega}^{\infty} e^{-y/2} y^{(6\alpha-6k-7)/12} \mid \phi(y) \mid dy < \infty \qquad \ldots(3.2)$$

$$\int_{\omega}^{\infty} e^{-y/2} y^{(6\alpha-6k+5)/12} \mid d\phi(y) \mid < \infty \qquad \ldots(3.3)$$

A being some constant not necessarily the same at each occurrence.

4. Proof of the Theorem

On account of the nth Cesàro mean of order k of the series (1.1) at $x = 0$ [see Szegö (1967), p. 272], orthogonality property of Laguerre polynomials and the relations :

$$A_n^k = \binom{n+k}{n} \sim n^k/\Gamma(k+1), \; A_{n-1}^k = \frac{n}{n+k} A_n^k$$

and $L_n^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x) = L_n^{(\alpha-1)}(x)$, we have

$$\sigma_n^k(0) - \sigma_{n-1}^k(0) = \{\Gamma(\alpha+1)\}^{-1} \int_0^{\infty} e^{-y} y^{\alpha} \phi(y) \frac{n L_n^{(\alpha+k)}(y) - k L_{n-1}^{(\alpha+k+1)}(y)}{n A_n^k} dy$$

$$= M(n) - N(n), \text{ say.} \qquad \ldots(4.1)$$

To prove the theorem it will be sufficient to show that $\sum_n |M(n)|$ and $\sum_n |N(n)|$ are convergent, where

$$M(n) = \{\Gamma(\alpha+1)\, A_n^k\}^{-1} \int_0^\infty e^{-y} y^\alpha \phi(y)\, L_n^{(\alpha+k)}(y)\, dy \qquad \ldots(4.2)$$

and

$$N(n) = \{\Gamma(\alpha+1)\, nA_n^k\}^{-1} . k . \int_0^\infty e^{-y} y^\alpha \phi(y)\, L_{n-1}^{(\alpha+k+1)}(y)\, dy. \qquad \ldots(4.3)$$

Applying the differential equation for Laguerre polynomials [see Rainville (1960), p. 204, eqn. (1) of section 117] and the relation $dL_n^{(\alpha)}(x)/dx = -L_{n-1}^{(\alpha+1)}(x)$ and integrating by parts, we can put the integral in $M(n)$ as

$$\int_0^\infty e^{-y} y^\alpha L_n^{(\alpha+k)}(y)\, \phi(y)\, dy = \frac{k}{n} \int_0^\infty e^{-y} y^\alpha L_{n-1}^{(\alpha+k+1)}(y)\, \phi(y)\, dy$$

$$- \frac{1}{n} \int_0^\infty e^{-y} y^{\alpha+1} L_{n-1}^{(\alpha+k+1)}(y)\, d\phi(y)$$

$$= P(n) - Q(n), \text{ say}. \qquad \ldots(4.4)$$

We set

$$P(n) - Q(n) = \int_0^{c/n} + \int_{c/n}^{\omega} + \int_{\omega}^{n} + \int_n^\infty = I_1 + I_2 + I_3 + I_4. \qquad \ldots(4.5)$$

We see that $\phi(0) = 0$, so that from (3.1)

$$|\phi(y)| \leqslant \int_0^y |d\phi(u)| \leqslant A y^{\alpha+1} \qquad \ldots(4.6)$$

and, therefore, using (7.6.8) from Szegö (1967, p. 177), we obtain by applying mean-value theorem

$$|I_1| = O(n^{-1})\, O(n^{\alpha+k+1}) \left[\int_0^{c/n} e^{-y} y^\alpha |\phi(y)|\, dy + \int_0^{c/n} e^{-y} y^{\alpha+1} |d\phi(y)|\right]$$

$$= O(n^{\alpha+k}) \left[\int_0^\eta y^\alpha . A y^{\alpha+1} dy + \int_0^\eta y^{\alpha+1} |d\phi(y)|\right], (0 < \eta = b/n \leqslant c/n)$$

$$= O(n^{k-\alpha-2}). \qquad \ldots(4.7)$$

Again

$$| I_2 | = O(n^{(2\alpha+2k-3)/4}) \left[\int_{c/n}^{\omega'} y^{(2\alpha-2k-3)/4} \, . \, Ay^{\alpha+1} dy + \int_{c/n}^{\omega'} y^{(2\alpha-2k+1)/4} \right.$$

$$\left. | d\phi(y) | \right], \; (c/n < \omega' \leqslant \omega)$$

$$= O(n^{(2\alpha+2k-3)/4}) + O(n^{k-\alpha-2}), \quad \text{...(4.8)}$$

integrating by parts and arranging. Now, using first part in (8 91.2) from Szegö (1967, p. 240) and keeping in view the analysis done in I_2 together with the assumptions (3.2) and (3.3), we see that $| I_3 | = O(n^{(2\alpha+2k-3)/4})$. With the second part in (8.91.2) from Szegö (1967, p. 240) along with (3.2) and (3.3), we have by mean-value theorem

$$| I_4 | = O(n^{(6\alpha+6k-7)/12}) \left[\int_n^{\infty} e^{-y/2} y^{(2\alpha-2k-3)/4} \, | \phi(y) | \, dy \right.$$

$$\left. + \int_n^{\infty} e^{-y/2} y^{(2\alpha-2k+1)/4} \, | d\phi(y) | \right]$$

$$= O(n^{(2\alpha+2k-3)/4}). \quad \text{...(4.9)}$$

Therefore

$$\sum_n | M(n) | = O\left(\sum_{n=1}^{\infty} \frac{1}{n^{(3+2k-2\alpha)/4}} \right) + O\left(\sum_{n=1}^{\infty} \frac{1}{n^{\alpha+2}} \right)$$

$$= O(1), \text{ for } k > \alpha + \tfrac{1}{2}, \text{ and } \alpha > -1. \quad \text{...(4.10)}$$

To consider the integral in $N(n)$, we proceed as in the integral of $M(n)$ and, therefore

$$\int_0^{\infty} e^{-y} y^{\alpha} L_{n-1}^{(\alpha+k+1)}(y) \, \phi(y) \, dy = \frac{(k+1)}{n} \int_0^{\infty} e^{-y} y^{\alpha} L_{n-2}^{(\alpha+k+2)}(y) \, \phi(y) \, dy$$

$$- \frac{1}{n-1} \int_0^{\infty} e^{-y} y^{\alpha+1} L_{n-2}^{(\alpha+k+2)}(y) \, d\phi(y) \quad \text{...(4.11)}$$

which can be treated on the lines of the integrals in (4.4) to give

$$\sum_n | N(n) | = O\left(\sum_{n=1}^{\infty} \frac{1}{n^{(2k-2\alpha+5)/4}} \right) + O\left(\sum_{n=1}^{\infty} \frac{1}{n^{\alpha+2}} \right)$$

$$= O(1). \quad \text{...(4.12)}$$

Thus, the theorem is proved.

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REFERENCES

Gupta, D. P. (1973). Cesàro summability of Laguerre series. *J. Approx. Theory*, **7**, 226-38.
Hille, E. (1926). On Laguerre series. *Proc. natn. Acad. Sci., U.S.A.*, **12**, 265-69; 348-52.
Rainville, E. D. (1960). Special Functions. MacMillan Co., New York.
Szegö, G. (1967). Orthogonal Polynomials. Am. Math. Soc. Colloq. Publ., New York.