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ALMOST PARACONTACT 3-STRUCTURE ON A DIFFERENTIABLE MANIFOLD

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In the present paper, we have defined almost paracontact 3-structure on a differentiable manifold, and established the existence and some of its properties. Existence of a Riemannian metric on such a manifold is also shown and studied some of its properties.

1. INTRODUCTION

The notion of almost paracontact structure on a differentiable manifold has been given by Sato (1976). Sharfuddin and Hussain (1977) have studied the para-contact metric and para-Sasakian structures on such a manifold analogous to K-contact and Sasakian structures on almost contact manifold.

In the present paper, we define almost paracontact 3-structure on a differentiable manifold and study some of its properties.

An n-dimensional differentiable manifold M is said to have an almost para-contact structure (ϕ, ξ, η), where ϕ is (1, 1) tensor field, ξ is a vector field and η is a 1-form defined on M such that

$$\phi^2 = I - \eta \otimes \xi, \phi(\xi) = 0, \eta(\xi) = 1 \text{ and } \eta \circ \phi = 0. \qquad ...(1.1)$$

A differentiable manifold M with almost paracontact structure (ϕ, ξ, η) is called an almost paracontact manifold. It is well-known that there exists a Riemannian metric g on an almost paracontact manifold $M(\phi, \xi, \eta)$, which satisfies

$$g(X, Y) = g(\phi X, \phi Y) + \eta(X)\,\eta(Y), g(X, \xi) = \eta(X), X, Y \in x(M) \qquad ...(1.2)$$

where $x(M)$ is the lie-algebra of vector fields on M. An almost paracontact manifold M with such a metric is called an almost paracontact metric manifold.

Let $M(\phi, \xi, \eta, g)$ be an almost paracontact metric manifold. Then it is said to be paracontact metric manifold if

$$(\nabla_X \xi = - \phi X, X \in x(M) \qquad ...(1.3)$$

and is said to be para-Sasakian manifold if

$$(\nabla_X \phi)(Y) = \eta(Y) X - 2\eta(X)\,\eta(Y)\,\xi + g(X, Y)\,\xi, X, Y \in x(M) \qquad ...(1.4)$$

where ∇ is Riemannian connection of g.

It is well-known that every para-Sasakian manifold is paracontact metric manifold.

2. Almost Paracontact 3-Structure

Let $M(\phi, \xi, \eta)$ be an almost paracontact manifold, and let $\mu \neq 0$ be a (1, 1) tensor field defined on M. If we put

$$\phi' X = \mu^{-1}\phi\mu X,\ \eta'(X) = \eta(\mu X) \text{ and } \mu\xi' = \xi \quad \text{...(2.1)}$$

then we observe that (ϕ', ξ', η') is also an almost paracontact structure defined on M, thereby indicating that an almost paracontact structure on a differentiable manifold is not unique. This leads us to define, what is called an almost paracontact 3-structure on a differentiable manifold.

Suppose a differentiable manifold M admits three almost paracontact structures (ϕ_i, ξ_i, η_i), $i = 1, 2,$ satisfying

$$\left.\begin{aligned} &\eta_i(\xi_j) = \eta_j(\xi_i) = 0 \\ &\phi_i(\xi_j) = \phi_j(\xi_i) = \xi_K \\ &\eta_i \circ \phi_j = \eta_j \circ \phi_i = \eta_K \\ \text{and} \quad &\phi_i\phi_j + \eta_j \otimes \xi_i = \phi_j\phi_i + \eta_i \otimes \xi_j = \phi_k \end{aligned}\right\} \quad \text{...(2.2)}$$

for a cyclic permutation (i, j, k) of (1, 2, 3), then M is said to have an almost paracontact 3-structure.

Theorem 2.1 — If a differentiable manifold M admits two almost paracontact structures (ϕ_i, ξ_i, η_i), $i = 1, 2,$ satisfying

$$\left.\begin{aligned} &\eta_1(\xi_2) = \eta_2(\xi_1) = 0 \\ &\phi_1(\xi_2) = \phi_2(\xi_1) \\ &\eta_1 \circ \phi_2 = \eta_2 \circ \phi_1 \\ \text{and} \quad &\phi_1\phi_2 + \eta_2 \otimes \xi_1 = \phi_2\phi_1 + \eta_1 \otimes \xi_2 \end{aligned}\right\} \quad \text{...(2.3)}$$

then it admits an almost paracontact 3-structure.

Proof : Define a triplet (ϕ_3, ξ_3, η_3) on M by

$$\phi_3 = \phi_1\phi_2 + \eta_2 \otimes \xi_1,\ \xi_3 = \phi_1(\xi_2) \text{ and } \eta_3 = \eta_1 \circ \phi_2 \quad \text{...(2 4)}$$

It can be easily shown that (ϕ_3, ξ_3, η_3) is also an almost paracontact structure on M. Further using (2.3) and (2.4) we can get (2.2).

Theorem 2.2 — Suppose a differentiable manifold M admits two almost paracontact structures (ϕ_i, ξ_i, η_i), $i = 1, 2,$ and let there be given a Riemannian metric on M associated to both the structures and if

$$\phi_1\phi_2 + \eta_2 \otimes \xi_1 = \phi_2\phi_1 + \eta_1 \otimes \xi_2$$

then

(a) $\eta_1(\xi_2) = \eta_2(\xi_1) = 0$

(b) $\phi_1(\xi_2) = \phi_2(\xi_1)$

(c) $\eta_1 \circ \phi_2 = \eta_2 \circ \phi_1$.

PROOF : Since g is associated metric for both the structures we have

$$g(\xi_1, \xi_2) = \eta_1(\xi_2) = \eta_2(\xi_1). \quad \text{...(2.5)}$$

Using the given condition, we have

$$g(\phi_1\phi_2X + \eta_2(X)\,\xi_1, Y) = g(\phi_2\phi_1X, Y) + \eta_1(X)\,\eta_2(Y)$$

i.e. $$g(\phi_2X, \phi_1Y) + \eta_2(X)\,\eta_1(Y) = g(\phi_1X, \phi_2Y) + \eta_1(X)\,\eta_2(Y).$$

Put $X = \xi_1$ and $Y = \xi_2$ in the above equation and using (2.5) we get

$$\begin{aligned} g(\phi_2\xi_1, \phi_1\xi_2) &= g(\xi_1, \xi_1) - g(\xi_1\xi_2)\,\eta_1(\xi_2) \\ &= g(\xi_1, \xi_1 - \eta_1(\xi_2)\,\xi_2) \\ &= g(\xi_1, \xi_1 - \eta_2(\xi_1)\,\xi_2) \\ &= g(\xi_1, \phi_2^2\,\xi_1) \\ &= g(\phi_2\xi_1, \phi_2\xi_1). \end{aligned} \quad \text{...(2.6)}$$

This gives $\phi_1\xi_2 = \phi_2\xi_1$.

Using (2.6) we get

$$\begin{aligned} \eta_1(\phi_2X) &= g(\xi_1, \phi_2X) = g(\phi_2\xi_1, X) = g(\phi_1\xi_2, X) \\ &= g(\xi_2, \phi_1X) = \eta_2(\phi_1X). \end{aligned} \quad \text{...(2.7)}$$

Hence $\eta_1 \circ \phi_2 = \eta_2 \circ \phi_1$.

Lastly using (2.5), (2.6) and (2.7) we have

$$\begin{aligned} \phi_1(\xi_2) = \phi_2(\xi_1) &= \phi_2(\xi_1) - \eta_1(\phi_1\xi_2)\,\xi_1 \\ &= \phi_2(\xi_1) - \eta_1(\phi_2\xi_1)\,\xi_1 \\ &= \phi_1^2\,(\phi_2(\xi_1) \end{aligned}$$

which gives

$$\begin{aligned} \xi_2 &= \phi_1\phi_2\xi_1 \\ &= \phi_1\phi_1\xi_2 \\ &= \xi_2 - \eta_1(\xi_2)\,\xi_1. \end{aligned}$$

Hence $\eta_1(\xi_2)\,\xi_1 = 0$, giving $\eta_1(\xi_2) = 0$ and by (2.5) we have

$$\eta_1(\xi_2) = \eta_2(\xi_1) = 0$$

completing proof of the theorem.

As a consequence of last two theorems we have the following :

Corollary 2.1 — If a differentiable manifold M admits two almost paracontact metric structures $(\phi_1, \xi_1, \eta_1, g)$ and $(\phi_2, \xi_2, \eta_2, g)$ together with the condition

$$\phi_1\phi_2 + \eta_2 \otimes \xi_1 = \phi_2\phi_1 + \eta_1 \otimes \xi_2$$

then it admits an almost paracontact 3-structure.

Lemma 2.1 — If (ϕ_i, ξ_i, η_i), $i = 1, 2, 3$ is an almost paracontact 3-structure on a differentiable manifold M then vectors ξ_1, ξ_2 and ξ_3 are linearly independent

PROOF : Let $a_i \in R$ be such that

$$a_1\xi_1 + a_2\xi_2 + a_3\xi_3 = 0.$$

Applying ϕ_1 and then ϕ_2 on the above equation and using (2.2) we get

$$a_2 = 0.$$

Further applying ϕ_1, we get $a_3 = 0$, and hence $a_1 = 0$.

Therefore the vectors ξ_1, ξ_2 and ξ_3 are linearly independent.

3. ASSOCIATED RIEMANNIAN METRIC OF ALMOST PARACONTACT 3-STRUCTURE

In this section we establish the existence of a Riemannian metric on a differentiable manifold with an almost paracontact 3-structure, analogous to the associated metric of an almost paracontact structure.

Theorem 3.1 — For an almost paracontact 3-structure (ϕ_i, ξ_i, η_i), $i = 1, 2, 3$ on a differentiable manifold M, there exists a Riemannian metric g such that

$$g(X, \xi_i) = \eta_i(X),\ i = 1, 2, 3\ X \in x(M).$$

PROOF : Let g_1 be the associated Riemannian metric to (ϕ_1, ξ_1, η_1) and define a metric g_2 by

$$g_2(X, Y) = g_1(X - \eta_2(X)\,\xi_2,\ Y - \eta_2(Y)\,\xi_2) + \eta_2(X)\,\eta_2(Y).$$

Now define g by

$$g(X, Y) = g_2(X - \eta_3(X)\,\xi_3,\ Y - \eta_3(Y)\,\xi_3) + \eta_3(X)\,\eta_3(Y).$$

Then clearly g is Riemannian metric defined on M, and we have

$$\begin{aligned} g(X, \xi_1) &= g_2(X - \eta_3(X)\,\xi_3, \xi_1) \\ &= g_1(X - \eta_3(X)\,\xi_3 - \eta_2(X - \eta_3(X)\,\xi_3)\,\xi_2, \xi_1) \\ &= g_1(X, \xi_1) - \eta_3(X)\,g_1(\xi_3, \xi_1) - \eta_2(X)\,g_1(\xi_2, \xi_1) \\ &= \eta_1(X). \end{aligned}$$

Further we have

$$g(X, \xi_2) = g_2(X - \eta_3(X)\,\xi_3, \xi_2) = \eta_2(X)$$

and

$$g(X, \xi_3) = \eta_3(X).$$

This completes the proof of the theorem.

Theorem 3.2 — In a differentiable manifold M with an almost paracontact 3-structure (ϕ_i, ξ_i, η_i), $i = 1, 2, 3$; there exists a Riemannian metric, which is an associated metric for each of the three almost paracontact structures.

PROOF : Let h be the Riemannian metric of Theorem 3.1, then define a metric g on M by

$$g(X, Y) = \tfrac{1}{4}\,[h(X, Y) + \sum_{i=1}^{3} (h(\phi_i X, \phi_i Y) + \eta_i(X)\eta_i(Y))].$$

We observe that

$$\begin{aligned} g(X, \xi_1) &= \tfrac{1}{4}\,[h(X, \xi_1) + h(\phi_2 X, \phi_2 \xi_1) + h(\phi_3 X, \phi_3 \xi_1) + \eta_1(X)] \\ &= \tfrac{1}{4}\,[2\eta_1(X) + h(\phi_2 X, \xi_3) + h(\phi_3 X, \xi_2)] \\ &= \tfrac{1}{4}\,[2\eta_1(X) + \eta_3(\phi_2 X) + \eta_2(\phi_3 X)] = \eta_1(X). \end{aligned}$$

Similarly we have

$$g(X, \xi_2) = \eta_2(X) \text{ and } g(X, \xi_3) = \eta_3(X).$$

Now let us calculate $g(\phi_1 X, \phi_1 Y)$, for this we have

$$\begin{aligned} g(\phi_1 X, \phi_1 Y) = \tfrac{1}{4}\,[h(\phi_1 X, \phi_1 Y) &+ h(\phi_1^2 X, \phi_1^2 Y) + h(\phi_2\phi_1 X, \phi_2\phi_1 Y) \\ &+ h(\phi_3\phi_1 X, \phi_3\phi_1 Y) + \eta_2(\phi_1 X)\,\eta_2(\phi_1 Y) \\ &+ \eta_3(\phi_1 X)\,\eta_3(\phi_1 Y)]. \end{aligned}$$

Using (2.2) and Theorem 3.1 we get

$$\begin{aligned} g(\phi_1 X, \phi_1 Y) &= \tfrac{1}{4}\,[h(X, Y) + \sum_{i=1}^{3} (h(\phi_i X, \phi_i Y) + \eta_i(X)\,\eta_i(Y)) \\ &\qquad - 4\eta_1(X)\,\eta_1(Y)] \\ &= g(X, Y) - \eta_1(X)\,\eta_1(Y). \end{aligned}$$

Similarly we can show that

$$g(\phi_2 X, \phi_2 Y) = g(X, Y) - \eta_2(X)\,\eta_2(Y)$$

and

$$g(\phi_3 X, \phi_3 Y) = g(X, Y) - \eta_3(X)\,\eta_3(Y).$$

The metric defined above is called an associated metric of almost paracontact 3-structure.

Lemma 3.1 — In a differentiable manifold M with almost paracontact 3-structure (ϕ_i, ξ_i, η_i) and associated metric g we have

$$g(\phi_i X, \phi_j Y) = g(\phi_k X, Y) - \eta_i(X)\,\eta_j(Y).$$

4. Paracontact and Para-sasakian 3-Structures

Lemma 4.1 — If a differentiable manifold M with almost paracontact 3-structure (ϕ_i, ξ_i, η_i), $i = 1, 2, 3$ and associated metric g, admits a paracontact metric 3-structure, then

(a) $(\nabla_{\xi_1}\phi_1)(\xi_2) = (\nabla_{\xi_3}\phi_1)(\xi_2) = 0$

(b) $(\nabla_{\xi_2}\phi_2)(\xi_1) = (\nabla_{\xi_3}\phi_2)(\xi_1) = 0$

(c) $(\nabla_{\xi_3}\phi_3)(\xi_2) = (\nabla_{\xi_1}\phi_3)(\xi_2) = 0$

where ∇ is Riemannian connection of g.

Proof : We have on using (1.3) and (2.2),

$$\nabla_X \xi_3 = \nabla_X \phi_1 \xi_2$$
$$= (\nabla_X \phi_1)(\xi_2) + \phi_1(\nabla_X \xi_2)$$

i.e.

$$-\phi_3 X = (\nabla_X \phi_1)(\xi_2) - \phi_1\phi_2(X).$$

Hence

$$(\nabla_X \phi_1)(\xi_2) = -\eta_2(X)\,\xi_1.$$

Putting $X = \xi_1$ and ξ_3 we get (a) and the proof of the rest is similar.

Theorem 4.1 — If M is a differentiable manifold with almost paracontact 3-structure (ϕ_i, ξ_i, η_i), $i = 1, 2, 3$ and associated metric g, then M cannot admit para-Sasakian 3-structure.

Proof : Suppose M admits para-Sasakian 3-structure, then we have from last theorem

$$(\nabla_X \phi_1)(\xi_2) = -\eta_2(X)\,\xi_1.$$

Using (1.4) in above equation we get

$$g(X, \xi_2)\, \xi_1 = -\, \eta_2(X)\, \xi_1$$

or

$$\eta_2(X)\, \xi_1 = -\, \eta_2(X)\, \xi_1$$

which is contradiction. Hence M cannot admit para-Sasakian 3-structure.

5. An Example

Consider R^3, and define

$$\phi_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \phi_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \phi_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\xi_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \xi_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \xi_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

and

$$\eta_1 = [0, 1, 0], \quad \eta_2 = [1, 0, 0], \quad \eta_3 = [0, 0, 1]$$

then (ϕ_i, ξ_i, η_i) is an almost paracontact 3-structure on R^3.

The associated Riemannian metric of this structure is given by

$$g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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