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# A FIXED POINT THEOREM IN METRIC SPACES

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A fixed point theorem of a continuous mapping of a compact metric space into itself has been proved in this note.

Recently Dass and Gupta (1975) have proved the following theorem.

*Theorem A* — Let $f$ be a mapping of a metric space $(X, d)$ into itself such that

$$\text{(i)} \quad d(f(x), f(y)) \leqslant \frac{\alpha d(y, f(y))\,[1 + d(x, f(x))]}{1 + d(x, y)} + \beta d(x, y)$$

for all $x, y \in X$, $\alpha, \beta > 0$, $\alpha + \beta < 1$ and

(ii) for some $x_0 \in X$, the sequence of iterates $\{f^n(x_0)\}$ has a subsequence $\{f^{n_k}(x_0)\}$ with

$$\xi = \lim_{n \to \infty} f^{n_k} x_0.$$

Then $\xi$ is a unique fixed point of $f$.

Now we will prove the following theorem :

*Theorem* 1 — If $F$ is a continuous mapping of a compact metric space $(X, d)$ into itself satisfying the condition,

$$d(F(x), F(y)) \leqslant \frac{\alpha d(y, F(y))\,[1 + d(x, F(x))]}{1 + d(x, y)} + \beta d(x, y)$$

for all $x, y \in X$, $\alpha, \beta > 0$, $\alpha + \beta = 1$, then $F$ has a unique fixed point.

PROOF : First we define a function $T$ on $X$ as follows :

$$T(x) = d(x, F(x)), \text{ for all } x \in X.$$

Since $d$ and $F$ are continuous on $X$, $T$ is also continuous on $X$.

From compactness of $X$ there exists a point $\xi \in X$ such that

$$T(\xi) = \inf \{T(x); x \in X\}. \quad \ldots(1)$$

If $T(\xi) \neq 0$, it follows that, $\xi \neq F\xi$ and so,

$$T(F(\xi)) = d(F(\xi), F^2(\xi))$$

$$\leqslant \frac{\alpha d(F(\xi), F^2(\xi))\,[1 + d(\xi, F(\xi)]}{1 + d(\xi, F(\xi))} + \beta d(\xi, F(\xi))$$

$$\leqslant \alpha d(F(\xi), F^2(\xi)) + \beta d(\xi, F(\xi))$$

$$\therefore \quad d(F(\xi), F^2(\xi)) < \frac{\beta}{1 - \alpha} d(\xi, F(\xi)) = d(\xi, F(\xi))$$

which implies, $T(F(\xi)) < T(\xi)$, which is a contradiction to the condition (1) and hence $\xi = F(\xi)$. Consequently $\xi$ is a fixed point of $F$.

Now we will prove the uniqueness of $\xi$. Let, if possible, $\eta \neq \xi$ be another fixed point of $F$.

Now $$d(\xi, \eta) = d(F(\xi), F(\eta))$$

$$\leqslant \frac{\alpha d(\eta, F(\eta))\,[1 + d(\xi, F(\xi))]}{1 + d(\xi, \eta)} + \beta d(\xi, \eta)$$

i.e. $$d(\xi, \eta) \leqslant \beta d(\xi, \eta) < d(\xi, \eta)$$

which is a contradiction as $\beta < 1$. $\therefore$ $\xi = \eta$

Hence $\xi$ is a unique fixed point of $F$.

Hence the theorem.


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## REFERENCE


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