Indian J. pure appl Math., **10**(4) : 451-456, April 1979

ON THE STRONG (J, p_n)-SUMMABILITY OF FOURIER SERIES

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(*Received* 3 *August* 1978; *after revision* 7 *October* 1978)

Khan (1972) obtained a criterion for ordinary (J, p_n) summability of Fourier series. In the present paper, we have defined for the first time strong (J, p_n)-summability and applied it to Fourier series.

§1. Let $p_n > 0$ be such that $\sum_{n=0}^{\infty} p_n$ diverges and the radius of convergence of the power series

$$p(x) = \sum_{n=0}^{\infty} p_n x^n \qquad \ldots(1.1)$$

is 1. Given any series Σu_n with the sequence of partial sums $\{s_n\}$, we write

$$p_s(x) = \sum_{n=0}^{\infty} p_n s_n x^n \qquad \ldots(1.2)$$

and

$$J_s(x) = \frac{p_s(x)}{p(x)}. \qquad \ldots(1.3)$$

If the series on the right of (1.2) is convergent in the right open interval [0, 1), and if $J_s \to s$, as $x \to 1 - 0$, we say that the series Σu_n or the sequence $\{s_n\}$ is summable (J, p_n) to the sum s, where s is finite (Borwein 1957; Hardy 1949, p. 80).

If $$\sum_{n=0}^{\infty} p_n \mid s_n - s \mid x^n = o(p(x)), \text{ as } x \to 1 - 0$$

then we say that the series Σu_n with the sequence of partial sums $\{s_n\}$ is strongly summable (J, p_n) or simply summable $[J, p_n]$ to the sum s.

For $p_n = \dfrac{1}{n+1}$, (J, p_n) and $[J, p_n]$ summabilities are reduced to (L) (Borwein 1958) and $[L]$ (Rai 1966) summabilities respectively.

§2. Let $f(t) \in L(-\pi, \pi)$ and be periodic with period 2π. The Fourier series associated with $f(t)$ is

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t). \qquad \ldots(2.1)$$

We shall denote by $s_n(\theta)$ the nth partial sum of the series (2.1) at θ and write

$$\phi(t) = \tfrac{1}{2}\{f(\theta + t) + f(\theta - t) - 2s\}.$$

§3. In this paper, we prove the following :

Theorem — Let the sequence $\{p_n\}$ be positive and decreasing steadily to zero, such that $\{np_n\}$ is bounded. If

$$\int_0^t |\phi(u)|\, du = o(tp(1 - t))\ (t \to +0), \qquad \ldots(3.1)$$

$$\int_t^\delta \left|\frac{\phi(u)}{u}\right| \log^+ \left|\frac{\phi(u)}{u}\right| du = o(p(1 - t))^* \qquad \ldots(3.2)$$

as $t \to +0$, for any arbitrary δ, $0 < \delta < \pi$, then

$$\sum_{n=0}^{\infty} |s_n(\theta) - s|\, p_n x^n = o(p(x)), \qquad \ldots(3.3)$$

as $x \to 1 - 0$.

§4. We require the following lemmas :

Lemma 1 (Hardy and Littlewood 1935) — If

$$f_R(t) = \sum_{-R}^{R} C_n e^{nit} \quad \text{and} \quad f_R^+(t) = \sum_{-R}^{R} C_n^+ e^{nit}$$

where C_n^+ are the numbers $|C_n|$ rearranged in an order such that

$$C_0^+ \geqslant C_{-1}^+ \geqslant C_1^+ \geqslant C_{-2}^+ \geqslant C_2^+ \geqslant C_{-3}^+ \geqslant \ldots^{**}$$

then

$$\int_{-\pi}^{\pi} \exp(b\,|f_R(t)|)dt \leqslant 2 \int_{-\pi}^{\pi} \exp(b\,|f_R^+(t)|)dt$$

for every positive b.

Lemma 2 — If the sequence $\{p_n\}$ is positive and decreases steadily to zero, then

(i) $\sum_{n=0}^{\infty} p_n x^n \sin nt = O(1)$

*$\log^+ x$ denotes the function which is equal to $\log x$ for $x > 1$ and to zero elsewhere.

**This notation is due to Gabriel (1932).

and

$$\text{(ii)} \quad \sum_{n=0}^{\infty} p_n x^n \cos nt = O(1)$$

for $0 \leqslant x < 1$ and for all real t.

PROOF : Since the series $\Sigma\, x^n \sin nt$ and $\Sigma\, x^n \cos nt$ are both convergent for $0 \leqslant x < 1$ and for all real t, and the sequence $\{p_n\}$ is positive and decreases steadily to zero, both of the parts (i) and (ii) of the lemma immediately follow from Dirichlet's test for convergence of infinite series.

§5. *Proof of the Theorem* : It is well-known that

$$s_n(\theta) - s = \frac{2}{\pi} \int_0^{\pi} \frac{\sin (n + \frac{1}{2})\, u}{2 \sin \frac{1}{2} u}\, \phi(u)\, du$$

$$= \frac{2}{\pi} \int_0^{\pi} \frac{\sin nu}{u}\, \phi(u)\, du + o(1)$$

by Riemann Lebesgue theorem.

Therefore

$$\sum_{n=0}^{\infty} p_n x^n \mid s_n(\theta) - s \mid = \frac{2}{\pi} \sum_{n=0}^{\infty} p_n x^n \left| \int_0^{\pi} \frac{\phi(u)}{u} \sin nu\, du \right| + o(p(x))$$

$$= \frac{2}{\pi} \sum_{n=0}^{\infty} p_n x^n \left| \left(\int_0^{1-x} + \int_{1-x}^{\pi} \right) \frac{\phi(u)}{u} \sin nu\, du \right| + o(p(x))$$

$$\leqslant \frac{2}{\pi} \sum_{n=0}^{\infty} p_n x^n \mid I_1 \mid + \frac{2}{\pi} \sum_{n=0}^{\infty} p_n x^n \mid I_2 \mid + o(p(x)), \text{ say.} \quad ...(5.1)$$

Now

$$\sum_{n=0}^{\infty} p_n x^n \mid I_1 \mid = \sum_{n=0}^{\infty} p_n x^n \left| \int_0^{1-x} \frac{\phi(u)}{u} \sin nu\, du \right|$$

$$\leqslant \sum_{n=0}^{\infty} n p_n x^n \int_0^{1-x} \mid \phi(u) \mid du$$

(*equation continued on p.* 454)

$$= O\left(\frac{1}{1-x}\right) \int_0^{1-x} | \phi(u) | \, du,$$

by the hypothesis that $\{np_n\}$ is bounded.

$$= o(p(x)), \text{ by } (3.1). \qquad \ldots(5.2)$$

And

$$\sum_{n=0}^{\infty} p_n x^n \, | I_2 | = \int_{1-x}^{\pi} \frac{\phi(u)}{u} \sum_{n=0}^{\infty} \epsilon_n \sin nu \, p_n x^n \, du$$

where $\epsilon_n = \pm 1$ so as to make $\epsilon_n I_2 \geqslant 0$, where $n = 0, 1, 2, 3, \ldots$.

We have from Young's inequality (Hardy *et al.* 1952)

$$\omega v \leqslant l\omega \log^+ \omega + l e^{(v-l)/l} \text{ for } \omega > 0, \; l > 0 \qquad \ldots(5.3)$$

and for all real v.

Setting $\omega = \left|\frac{\phi(u)}{u}\right|, \; v = | \Omega(u) | = \left| \sum_{n=0}^{\infty} \epsilon_n \sin nu \, p_n x^n \right|$

and $l = 2$ in (5.3), we have from condition (3.2),

$$\left| \left(\sum_{n=0}^{\infty} p_n x^n \, | I_2 | \right) \right| = \left| \int_{1-x}^{\pi} \frac{\phi(u)}{u} \sum_{n=0}^{\infty} \epsilon_n \sin nu \, p_n x^n du \right|$$

$$\leqslant \int_{1-x}^{\pi} \left| \frac{\phi(u)}{u} \right| \left| \Omega(u) \right| du$$

$$\leqslant 2 \int_{1-x}^{\pi} \left| \frac{\phi(u)}{u} \right| \log^+ \left| \frac{\phi(u)}{u} \right| du + A^{\dagger} \int_{1-x}^{\pi} \exp\left(| \Omega(u) | /2 \right) du$$

$$= o(p(x)) + A \int_{1-x}^{\pi} \exp\left(| \Omega(u) | /2 \right) du$$

$$= o(p(x)) + K. \qquad \ldots(5.4)$$

Now, it remains to estimate K.

† A is a positive finite constant which is not necessarily the same at each occurrence.

Following Szász (1942, p.708), if we put $\epsilon_n p_n x^n = \sigma_n$, we have

$$\Omega_m(u) = I\,[r(u)] \text{ and } |\,\Omega_m(u)\,| \leqslant |\,r(u)\,|$$

where

$$r(u) = \sum_{n=0}^{m} \sigma_n e^{niu}$$

is a polynomial of degree m whose coefficients have (in some order of enumeration) the absolute values

$$p_0,\ p_1x,\ p_2x^2,\ p_3x^3,\ \ldots,\ p_mx^m.$$

Hence, in the notation of Lemma 1,

$$r^+(u) = p_0 + p_1xe^{-iu} + p_2x^2e^{iu} + p_3x^3e^{-2iu} + p_4x^4e^{2iu} + \ldots,$$

the last term having modulus p_mx^m and an argument depending upon the parity of m.

The real and imaginary parts of $r^+(u)$ are bounded for all m and $u(1-x \leqslant u \leqslant \pi)$, by Lemma 2(i) and 2(ii) respectively. Hence

$$|\,r^+(u)\,| \leqslant A. \qquad \ldots(5.5)$$

We now obtain from Lemma 1 and (5.5) that

$$\int_{1-x}^{\pi} \exp\,(\,|\,\Omega_m(u)\,|\,/2)\,du \leqslant A \int_{1-x}^{\pi} \exp\,(\,|\,r(u)\,|\,/2)\,du$$

$$\leqslant A \int_{1-x}^{\pi} \exp\,(\,|\,r^+(u)\,|\,/2)\,du$$

$$\leqslant A \int_{1-x}^{\pi} du \leqslant A$$

for all m.

Hence it follows that

$$K = A \int_{1-x}^{\pi} \exp\,(\,|\,\Omega(u)\,|\,/2)\,du \leqslant A. \qquad \ldots(5.6)$$

Collecting (5.4) and (5.6), we have

$$\sum_{n=0}^{\infty} p_nx^n\,|\,I_2\,| = o(p(x)). \qquad \ldots(5.7)$$

Finally, the collection of (5.1), (5.2) and (5.7) completes the proof of the theorem.

Acknowledgement

The authors are indebted to the referee for his valuable suggestions.

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