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ON INFINITE MATRICES AND INVARIANT MEANS

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Schaefer (1972) has introduced the concept of σ-conservative, σ-regular and σ-coercive matrices and obtained necessary and sufficient conditions to characterize these classes of matrices. The purpose of this paper is to characterize some more matrices in V_σ.

1. INTRODUCTION

Let l_∞ and c respectively be the Banach spaces of bounded and convergent sequences $x = \{x_k\}$ with the usual norm $\| x \| = \sup_k | x_k |$.

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional ϕ on l_∞ is said to be an invariant mean or σ-mean if and only if (i) $\phi(x) \geqslant 0$ when the sequence $x = \{x_n\}$ has $x_n \geqslant 0$ for all n; (ii) $\phi(e) = 1$, where $e = \{1, 1, \ldots\}$; and (iii) $\phi(\{x_{\sigma(n)}\}) = \phi(x)$ for all $x \in l_\infty$. Throughout this paper we deal only with mappings σ are one-to-one and are such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the mth iterate of the mapping at n. For such mappings, every σ-means extends the limit functional on c (Raimi 1963), in the sense that $\phi(x) = \lim x$ for all $x \in c$. Consequently, $c \subset V_\sigma$, where V_σ is the set of bounded sequences all of whose σ-means are equal. When $\sigma(n) = n + 1$, V_σ is the set of almost convergent sequences (Lorentz 1948).

Schaefer (1972) has obtained necessary and sufficient conditions to characterize (c, V_σ), $(c, V_\sigma)_{reg}$, and (l_∞, V_σ)-matrices. Quite recently Ahmad and Mursaleen (1979) have characterized the classes $(c(p), V_\sigma)$, $(c(p), V_\sigma)_{reg}$, and $(l_\infty(p), V_\sigma)$ which generalize the results of Schaefer (1972). In the present paper author obtains conditions which characterize $(l(p), V_\sigma)$, and $(M_0(p), V_\sigma)$ matrices.

2. PRELIMINARIES

If p_k is real such that $p_k > 0$ and $\sup_k p_k < \infty$, we define (Simons 1965 and Maddox 1967, 1969)

$$l_\infty(p) = \{x : \sup_k | x_k |^{p_k} < \infty\}$$

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$$c(p) = \{x : | x_k - l |^{p_k} \to 0 \text{ for some } l\}$$

$$l(p) = \{x : \sum_k | x_k |^{p_k} < \infty\}$$

and

$$M_0(p) = \bigcup_{N>1} \{x : \sum_k | x_k | N^{-1/p_k} < \infty\}$$

when $p_k = p \ \forall \ k$, we write $l_\infty(p) = l_\infty$, $c(p) = c$, $l(p) = l_p$ and $M_0(p) = l_1$, also $M_0(p) = l_1$ for inf $p_k > 0$.

If $x = \{x_n\}$, write $Tx = \{x_{\sigma(n)}\}$. The set V_σ can be characterized as the set of all bounded sequences x for which $\lim_m t_{mn}(x)$ exists in l_∞ and has the form Le, where $L = \sigma - \lim x$ and $t_{mn}(x) = \sum_{j=0}^{m} T^j x/(m + 1)$.

Throughout this paper we shall use the notation $a(n, k)$ to denote the element a_{nk} of the matrix A. We write for all integers $n, m \geqslant 1$

$$\begin{aligned} t_{mn}(Ax) &= (Ax + TAx + \dots + T^m Ax)/(m + 1) \\ &= \sum_k t(n, k, m)\, x_k \end{aligned}$$

where

$$t(n, k, m) = \frac{1}{m + 1} \sum_{j=0}^{m} a(\sigma^j(n), k).$$

3. Main Results

Theorem 3.1 — $A \in (l(p), V_\sigma)$ if and only if there exists $B > 1$ such that for every n

$$\text{(i)} \begin{cases} \sup_m \sum_k | t(n, k, m) |^{q_k} B^{-q_k} < \infty, \ (1 < p_k < \infty) \\ \sup_{m\,k} | t(n,k, m) |^{p_k} < \infty, \ (0 < p_k \leqslant 1) \end{cases}$$

(ii) $a_{(k)} = \{a_{nk}\}_{n=1}^{\infty} \in V_\sigma$ for each k

i.e. $\lim_m t(n, k, m) = u_k$ uniformly in n.

In this case, the σ-limit of A is

(iii) $\lim_m \sum_k t(n, k, m) = \sum_k u_k x_k.$

PROOF : Let $1 < p_k < \infty$ and $A \in (l(p), V_\sigma)$. Define $e_k = \{0, 0, \ldots, 0, 1(k\text{th place}), 0, \ldots\}$. Since $e_k \in l(p)$, therefore (ii) holds. Put $f_{mn}(x) = t_{mn}(Ax)$, since $t_{mn}(Ax)$ exists for each m and $x \in l(p)$. Therefore $\{f_{mn}(x)\}_m$ is a sequence of continuous real functionals on $l(p)$ and further $\sup_m |f_{mn}(x)| < \infty$ on $l(p)$. Now condition (i) follows by arguing with uniform boundedness principle. The case $0 < p_k \leqslant 1$ has similar proof.

Conversely, suppose that the conditions (i) and (ii) are true and $x \in l(p)$. Now we have for every $r \geqslant 1$

$$\sum_{k=1}^{r} |t(n, k, m)|^{q_k} B^{-q_k} \leqslant \sup_m \sum_{k=1}^{\infty} |t(n, k, m)|^{q_k} B^{-q_k}.$$

Therefore

$$\sum_k |u_k|^{q_k} B^{-q_k} = \lim_r \lim_m \sum_{k=1}^{r} |t(n, k, m)|^{q_k} B^{-q_k}$$

$$\leqslant \sup_m \sum_{k=1}^{\infty} |t(n, k, m)|^{q_k} B^{-q_k} < \infty.$$

Thus the series $\sum_k t(n, k, m)\, x_k$ and $\sum_k u_k x_k$ converge for each m and $x \in l(p)$. For a given $\epsilon > 0$ and $x \in l(p)$, choose k_0 such that

$$\left(\sum_{k=k_0+1}^{\infty} |x_k|^{p_k} \right)^{1/H} < \epsilon$$

where $H = \sup p_k$. Since (ii) holds, therefore there exists m_0 such that

$$\text{(iv)} \left| \sum_{k=1}^{k_0} (t(n, k, m) - u_k) \right| < \epsilon \ (\forall\, m > m_0).$$

By condition (ii) it follows that

$$\left| \sum_{k=k_0+1}^{\infty} (t(n, k, m) - u_k) \right|$$

is arbitrary small, therefore condition (iii) follows i.e.,

$$\lim_m \sum_k t(n. k, m)\, x_k = \sum_k u_k x_k$$

uniformly in n. This completes our proof.

Theorem 3.2 — $A \in (M_0(p), V_\sigma)$ if and only if for every integer $N > 1$

(i) $M_N = \sup_{m,k} |t(n, k, m)|\, N^{1/p_k} < \infty, (\forall\, n)$

(ii) $a_{(k)} \in V_\sigma$ for each k.

In this case, the σ-limit of Ax is same as in Theorem 3.1.

PROOF: Let $A \in (M_0(p), V_\sigma)$. Since $e_k \in M_0(p)$, (ii) must hold. Now on contrary suppose that (i) is not true then there exists $N > 1$ such that $M_N = \infty$. Therefore by Theorem 3.1, matrix $B = (b_{nk}) = (a_{nk}N^{1/p_k}) \notin (l_1, V_\sigma)$, i.e. there exists $x \in l_1$ such that $Bx \notin V_\sigma$. Now $y = (y_k) = (N^{1/p_k} x_k) \in M_0(p)$. But $Ay = Bx \notin V_\sigma$, which contradicts the fact that $A \in (M_0(p), V_\sigma)$. Hence (i) is true.

Conversely, suppose conditions (i) and (ii) are true and $x \in M_0(p)$. Then

$$\left| \sum_k t(n, k, m)\, x_k \right| \leqslant \sum_k | x_k | \, N^{-1/p_k} \, | t(n, k, m) | \, N^{1/p_k} < \infty.$$

Now arguing as in Theorem 3.1 we get

$$\lim_m \sum_k t(n, k, m)\, x_k = \sum_k u_k x_k.$$

Hence $A \in (M_0(p), V_\sigma)$.

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