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ON GENERAL DECOMPOSITION OF CURVATURE TENSOR FIELD $\underset{2}{K}{}^{i}_{jkh}$ IN A RECURRENT AREAL SPACE OF SECOND ORDER

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We discussed the decomposition of curvature tensor field $\underset{2}{K}{}^{i}_{jkh}$ in a recurrent Areal space of the second order. Certain relations and identities have been obtained in this paper.

1. INTRODUCTION

Let X_n be an n-dimensional differentiable manifold referred to local coordinates x^i. An m-dimensional subspace C_m $(m < n)$ can be represented parametrically in the form

$$x^i = x^i(t^\alpha)^* \qquad ...(1.1)$$

where t^α denotes a system of independent parameters in C_m. Assuming that the functions (1.1) are of class C^4, their first derivatives are denoted by

$$\dot{x}^i_\alpha = \frac{\partial x^i}{\partial t^\alpha}. \qquad ...(1.2)$$

We consider the Lagrange function L of the form

$$L = L(x^i, \dot{x}^i_\delta) \qquad ...(1.3)$$

satisfying the conditions :

(a) The Lagrangian L is of class C^4 in all its arguments and it is scalar with respect to the transformation of local coordinates x^i of X_n.

(b) The Lagrangian L is positive for all independent sets of arguments $\dot{x}^i_\alpha$.

*Throughout this paper the Latin indices i, j, k ... run over 1 to n while Greek indices $\alpha, \beta, \gamma, \epsilon$ ... run over 1 to m.

(c) The integral

$$I = \int_G L(x^i, \dot{x}^i_\varepsilon)\, dt^1 \ldots dt^m \qquad \ldots(1.4)$$

over a fixed region G of C_m is invariant under the transformations of the parameters t^α.

(d) The $nm \times nm$ determinant

$$D = \left| \frac{m}{2} \frac{\partial^2 L^{2/m}}{\partial \dot{x}^i_\alpha\, \partial \dot{x}^j_\beta} \right|$$

is non-vanishing for linearly independent $\dot{x}^i_\alpha$.

The Areal metric tensor is defined as (Rund 1966)

$$g^{\alpha\beta}_{ij} = \frac{m}{2} \frac{\partial^2 L^{2/m}}{\partial \dot{x}^i_\alpha\, \partial \dot{x}^j_\beta} \qquad \ldots(1.5)$$

which is symmetric in pairs of indices such as (α, i) and (β, j). The covariant derivative (Rund 1968) of a vector field $X^i_\varepsilon\,(x^h, \dot{x}^h_\varepsilon)$ w.r.t. x^j is given by

$$X^i_{\varepsilon\,|\,j} = \frac{\partial X^i_\varepsilon}{\partial x^j} - \frac{\partial X^i_\varepsilon}{\partial \dot{x}^l_\lambda} \Gamma^l_{pj}\, \dot{x}^p_\lambda + \Gamma^i_{jl}\, X^l_\varepsilon \qquad \ldots(1.6)$$

where Γ^i_{jl} is a connection coefficient of homogeneous degree zero in its directional arguments.

The commutation formula (Rund 1968) for a vector field $X^i_\varepsilon\,(x^h, \dot{x}^h_\varepsilon)$ involving the covariant derivative of the type (1.6) is given by

$$X^i_{\varepsilon\,|\,k\,|\,h} - X^i_{\varepsilon\,|\,h\,|\,k} = X^j_\varepsilon \underset{2}{K}{}^i_{jkh} - \frac{\partial X^i_\varepsilon}{\partial \dot{x}^l_\alpha} \underset{1}{K}{}^l_{pkh}\, \dot{x}^p_\alpha + X^i_{\varepsilon\,|\,j}\, T^j_{kh} \qquad \ldots(1.7)$$

where

$$\underset{1}{K}{}^l_{pkh} = - \underset{1}{K}{}^l_{phk} \qquad \ldots(1.8)$$

$$\underset{2}{K}{}^i_{jkh} = - \underset{2}{K}{}^i_{jhk} \qquad \ldots(1.9)$$

and

$$T^i_{kh} = \Gamma^i_{kh} - \Gamma^i_{hk} \qquad \text{...(1.10)}$$

are curvature tensors of first kind, second kind and torsion tensor respectively.

The curvature tensor of second kind satisfies the identity

$$\underset{2}{K}{}^i_{jkh} + T^i_{jk\,|\,h} + T^i_{lj}\, T^l_{kh} + \text{Cycl}\,(j, k, h) = 0, \qquad \text{...(1.11)}$$

where cycl (j, k, h) denotes the sum of two sets of three terms each obtained by replacing the indices (j, k, h) by first (k, h, j) and then by (h, j, k) in the expression on the left hand side of (1.11). The Bianchi identities (Rund 1968) corresponding to the curvature tensors of first and second kinds are given by

$$\underset{2}{K}{}^i_{jkh\,|\,l} + \frac{\partial \Gamma^i_{kj}}{\partial \dot{x}^p_\alpha}\, \underset{1}{K}{}^p_{qhl}\, \dot{x}^q_\alpha + T^q_{kh}\, \underset{2}{K}{}^i_{jql} + \text{cycl}\,(k, l, h) = 0 \qquad \text{...(1.12)}$$

where cycl (k, l, h) has the same meaning as in (1.11).

Definition 1.1 — An n-dimensional Areal space $A^{(m)}_n$ is said to be $\underset{2}{K}$-recurrent Areal space of first order, if the curvature tensor $\underset{2}{K}{}^i_{jkh}$ satisfies the relation (Pande and Dwivedi 1978a)

$$\underset{2}{K}{}^i_{jkh\,|\,l} = v_l\, \underset{2}{K}{}^i_{jkh}, \qquad \text{...(1.13)}$$

where v_l is a non-zero vector field.

Definition 1.2 — Areal space $A^{(m)}_n$ is said to be birecurrent areal space if the curvature tensor $\underset{2}{K}{}^i_{jkh}$ satisfies the condition (Pande and Dwivedi 1978a) :

$$\underset{2}{K}{}^i_{jkh\,|\,l\,|\,m} = a_{lm}\, \underset{2}{K}{}^i_{jkh}, \qquad \text{...(1.14)}$$

where $a_{lm} = a_{lm}(x^i, \dot{x}^i_s)$ is recurrent tensor field.

We shall denote such a space by $A^{(m)*}_n$.

2. Decomposition of $\underset{2}{K}{}^i_{jkh}$ in Recurrent Areal Space of Second Order

We consider the decomposion of $\underset{2}{K}{}^i_{jkh}\,(x^h, \dot{x}^h_s)$ as follows :

$$\underset{2}{K}{}^{i}_{jkh} = X^{i}_{j}\,\phi_{kh} \qquad \ldots(2.1)$$

and

$$X^{i}_{j}\,a_{ri} = d_{rj}, \qquad \ldots(2.2)$$

where X^{i}_{j}, ϕ_{kh} are tensor fields of homogeneous degree zero and a_{lm}, d_{rj} are recurrent tensor and decomposed tensor field respectively.

We shall use Theorem 3.1 of Pande and Dwivedi (1978b) which is as follows :

Theorem 2.1 — In a $\underset{2}{K}$-recurrent Areal space, if X^{i}_{j} is covariantly invariant then the tensor field ϕ_{kh} satisfies the recurrency condition i.e.

$$\phi_{kh\,|\,l} = v_l\phi_{kh}. \qquad \ldots(2.3)$$

Now we shall prove following theorems.

Theorem 2.2 — In $A_n^{(m)*}$, if X^{i}_{j} is covariantly invariant then the tensor field ϕ_{kh} satisfies the second order recurrency condition i.e.

$$\phi_{kh\,|\,l\,|\,m} = a_{lm}\phi_{kh}. \qquad \ldots(2.4)$$

PROOF : Differentiating (2.1) w.r.t. x^l covariantly and using the fact that X^{i}_{j} is covariantly invariant (i.e. $X^{i}_{j\,|\,l} = 0$), we get

$$\underset{2}{K}{}^{i}_{jkh\,|\,l} = X^{i}_{j}\,\phi_{kh\,|\,l}. \qquad \ldots(2.5)$$

Again differentiating (2.5) covariantly w.r.t. x^m and using the fact that X^{i}_{j} is covariantly invariant and eqn. (1.14), we get

$$a_{lm}\underset{2}{K}{}^{i}_{jkh} = X^{i}_{j}\,\phi_{kh\,|\,l\,|\,m}. \qquad \ldots(2.6)$$

With the help of (2.1), (2.6) yields the Theorem 2.2.

Theorem 2.3 — Under the decomposition (2.1), the decomposition tensor field ϕ_{kh} is skew symmetric in the indices k and h i.e.

$$\phi_{kh} = -\,\phi_{hk}. \qquad \ldots(2.7)$$

PROOF : With the help of (1.9) and (2.1), we get the relation (2.7).

Theorem 2.4 — In $A_n^{(m)*}$, the following relation holds under the decomposition (2.1) :

$$a_{lm}\phi_{kh} + a_{hm}\phi_{lk} + a_{km}\phi_{hl} = \phi_{lq}(T^q_{kh\,|\,m} + T^q_{kh}\, v_m)$$

$$+ \phi_{hq}\,(T^q_{lk\,|\,m} + T^q_{lk}\, v_m) + \phi_{kq}(T^q_{hl\,|\,m} + T^q_{hl}\, v_m). \qquad ...(2.8)$$

PROOF : Transvecting (1.12) by $\dot{x}^p_\alpha$ and using the fact that Γ^i_{lj} is homogeneous of degree zero and $\dot{x}^p_\alpha$ is non-zero, we obtain

$$\underset{2}{K}{}^i_{jkh\,|\,l} + \underset{2}{K}{}^i_{jlk\,|\,h} + \underset{2}{K}{}^i_{jhl\,|\,k} + T^q_{kh}\,\underset{2}{K}{}^i_{jql} + T^q_{lk}\,\underset{2}{K}{}^i_{jqh} + T^q_{hl}\,\underset{2}{K}{}^i_{jqk} = 0. \qquad ...(2.9)$$

Differentiating (2.9) covariantly w.r.t. x^m and using eqns. (1.13) and (1.14), we get

$$a_{lm}\underset{2}{K}{}^i_{jkh} + a_{hm}\underset{2}{K}{}^i_{jlk} + a_{km}\underset{2}{K}{}^i_{jhl} + T^q_{kh\,|\,m}\,\underset{2}{K}{}^i_{jql}$$

$$+ v_m\underset{2}{K}{}^i_{jql}\,T^q_{kh} + T^q_{lk\,|\,m}\,\underset{2}{K}{}^i_{jqh} + T^q_{lk}\,v_m\underset{2}{K}{}^i_{jqh}$$

$$+ T^q_{hl\,|\,m}\,\underset{2}{K}{}^i_{jqk} + T^q_{hl}\,v_m\underset{2}{K}{}^i_{jqk} = 0. \qquad ...(2.10)$$

Using eqns. (2.1) and (2.7) in (2.10), we obtain Theorem 2.4.

Theorem 2.5 — In $A^{(m)*}_n$, under the decomposition (2.1) the relation

$$(2a^{\dagger}_{[lm]} - v_s T^s_{lm})\,\phi_{kh} = (\phi_{hs}X^s_k + \phi_{sk}X^s_h)\,\phi_{lm} \qquad ...(2.11)$$

holds.

PROOF : Commuting (2.4) in l and m and using the commutation formula (1.7) and eqns. (2.1), (1.8), (2.7) and (2.3), we get

$$(a_{lm} - a_{ml})\,\phi_{kh} + \phi_{sh}X^s_k\,\phi_{lm} + \phi_{ks}X^s_h\,\phi_{lm}$$

$$+ v_s\phi_{hk}T^s_{lm} = \frac{\partial\phi_{kh}}{\partial\dot{x}^p_\alpha}\,\underset{1}{K}{}^p_{qml}\,\dot{x}^q_\alpha\,. \qquad ...(2.12)$$

Transvecting (2.12) by $\dot{x}^p_\alpha$ and using the fact that $\dot{x}^p_\alpha$ is non-zero and ϕ_{kh} is skew-symmetric homogeneous of degree zero in its directional arguments, we get the Theorem 2.5.

Theorem 2.6 — In $A^{(m)*}_n$, the identity for the torsion-tensor is given by

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† $2a_{[lm]} = a_{lm} - a_{ml}$.

$$a_{lm} d_{r\,[j}\phi^{**}_{kh]} = -\, a_{ri}(T^{i}_{[jk\,|\,h]\,|\,l\,|\,m} + T^{i}_{l\,[j}T^{l}_{kh]\,|\,l\,|\,m} \qquad \text{...(2.13)}$$

under the decompositions (2.1) and (2.2).

PROOF : Differentiating (1.11) covariantly w.r.t. x^l and x^m and simplifying the result in the view of eqns. (1.14) and (2.1), we obtain

$$a_{lm} X^{i}_{[j}\phi_{kh]} = -\, T^{i}_{[jk\,|\,h]\,|\,l\,|\,m} - T^{i}_{l[j}T^{l}_{kh]\,|\,l\,|\,m}. \qquad \text{...(2.14)}$$

Multiplying (2.14) by a_{ri} and using the relation (2.2), we get the Theorem 2.6.

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$^{**}d_{r[j}\phi_{kh]} = d_{rj}\phi_{kh} + d_{rk}\phi_{hj} + d_{rh}\phi_{jk}.$