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ON INDUCED THEORY OF SEMI-SYMMETRIC AND QUARTER-SYMMETRIC LINEAR CONNECTIONS

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In a recent paper Golab (1975) has developed the theory of semi-symmetric and quarter-symmetric linear connection parameters. In this paper we introduce the connection parameters induced by the linear connection of Golab, in a hypersurface of the given space. The conditions under which the induced connection is semi-symmetric or quarter-symmetric are investigated. Several properties of this connection parameter are studied.

1. INTRODUCTION

Let L_n be an n dimensional space equipped with a coordinate system x^i and endowed with a linear connection Γ^i_{jk}. It is assumed that the torsion tensor

$$S^i_{jk} \overset{def}{=} \Gamma^i_{[jk]} \overset{def}{=} \tfrac{1}{2}(\Gamma^i_{jk} - \Gamma^i_{kj}) \qquad ...(1.1)$$

is, in general, non-vanishing. Golab (1975) has given the following definitions :

Definition 1.1 — The connection parameter Γ^i_{jk} is called semi-symmetric, or Γ^*, if and only if there exists a field of covariant vectors S_i such that

$$S^i_{jk} = \delta^i_{[j} S_{k]} \qquad ...(1.2)$$

where δ^i_j are Kronecker deltas.

Definition 1.2 — The connection Γ^i_{jk} is called special semi-symmetric or Γ^{**}, if and only if the condition (1.2) holds and $S_i = \partial_i\sigma$ is gradient of a scalar σ.

Definition 1.3 — The connection parameter Γ^i_{jk} is said to be quarter-symmetric if and only if there exists an arbitrary tensor field t^i_j and a vector field S_i such that

$$S^i_{jk} = t^i_{[j} S_{k]}. \qquad ...(1.3)$$

2. Induced Connection Parameter

Consider a hypersurface characterized by the equations

$$x^i = x^i(u^\alpha), \ (i = 1, 2, \ldots, n; \alpha = 1, 2, \ldots, n-1).$$

The components X^i and U^α, of a vector field (of the hypersurface), in the imbedding space L_n and hypersurface L_{n-1} are related by

$$X^i = B^i_\alpha \, U^\alpha, \quad \text{where } B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}. \qquad \ldots(2.1)$$

The connection Γ^i_{jk} may be used in defining the absolute differential

$$DX^i = dX^i + \Gamma^i_{jk} X^j dx^k. \qquad \ldots(2.2)$$

The induced differential in the hypersurfce is defined by

$$DU^\alpha = B^\alpha_i \, DX^i \qquad \ldots(2.3)$$

where B^α_i is the inverse of the projection factor B^i_α. In other words

$$B^i_\alpha \, B^\beta_i = \delta^\beta_\alpha. \qquad \ldots(2.4)$$

Putting

$$DU^\alpha = dU^\alpha + B^i_{\alpha\beta} \, U^\alpha dU^\beta, \ dx^k = B^k_\gamma \, du^\gamma, \ B^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}$$

and simplifying with the help of eqns. (2.1), (2.2), (2.3) and the relations

$$dX^i = B^i_\alpha \, dU^\alpha + B^i_{\alpha\beta} \, U^\alpha du^\beta, \ dx^k = B^k_\gamma \, du^\gamma$$

we get

$$\Gamma^\alpha_{\beta\gamma} = B^\alpha_i \, (B^i_{\beta\gamma} + \Gamma^i_{jk} \, B^j_\beta \, B^k_\gamma). \qquad \ldots(2.5)$$

Adopting the usual terminology we shall call $\Gamma^\alpha_{\beta\gamma}$ as the connection parameter induced by Γ^i_{jk}. In view of the relations (1.1) and (2.5) the torsion tensor of the induced connection is given by

$$S^\alpha_{\beta\gamma} \overset{def}{=} \Gamma^\alpha_{[\beta\gamma]} = B^\alpha_i \, S^i_{jk} \, B^j_\beta \, B^k_\gamma. \qquad \ldots(2.6)$$

Suppose that the connection Γ^i_{jk} is semi-symmetric. Using eqns. (1.2), (2.4) and (2.6) we get

$$S^{\alpha}_{\beta\gamma} = \delta^{\alpha}_{[\beta} S_{\gamma]} \qquad \ldots(2.7)$$

where

$$S_\gamma = B^i_\gamma S_i \qquad \ldots(2.8)$$

This proves the following theorem :

Theorem 2.1 — The induced connection parameter $\Gamma^{\alpha}_{\beta\gamma}$ is semi-symmetric provided that Γ^{i}_{jk} is semi-symmetric.

Further we have the following :

Therem 2.2 — If the connection parameter Γ^{i}_{jk} is Γ^{**}, then $\Gamma^{\alpha}_{\beta\gamma}$ is also Γ^{**}.

The proof of this theorem follows immediately from Theorem 2.1 and the fact

$$S_\gamma = S_i B^i_\gamma = \partial_i \sigma \frac{\partial x^i}{\partial u^\gamma} = \partial_\gamma \sigma.$$

We now proceed to establish the following theorem :

Theorem 2.3 — The induced connection $\Gamma^{\alpha}_{\beta\gamma}$ is quarter-symmetric provided that Γ^{i}_{jk} is quarter-symmetric.

PROOF : Using eqns. (1.3), (2.6) and (2.8) we get

$$S^{\alpha}_{\beta\gamma} = t^{\alpha}_{[\beta} S_{\gamma]} \qquad \ldots(2.9)$$

where

$$t^{\alpha}_{\beta} = t^{i}_{j} B^{\alpha}_{i} B^{j}_{\beta}. \qquad \ldots(2.10)$$

This proves the theorem.

For the connection parameter Γ^{i}_{jk}, $\Gamma^{\alpha}_{\beta\gamma}$ we define vector fields

$$T_k = \Gamma^{i}_{ik} \text{ and } T_\gamma = \Gamma^{\alpha}_{\alpha\gamma}$$

which in accordance with the terminology of Vranceami (1957) will be called the torsion vectors of Γ^{i}_{jk} and $\Gamma^{\alpha}_{\beta\gamma}$ respectively.

Assuming that Γ^{i}_{jk} is semi-symmetric and using the relations (1.2) and (2.7) we get

$$T_k = (n - 1) S_k, \; T_\gamma = (n - 2) S_\gamma \qquad ...(2.11)$$

which, in view of eqn. (2.8), yields

$$T_\gamma = \frac{n-2}{n-1} T_k B^k_\gamma. \qquad ...(2.12)$$

This shows that the torsion vector of a hypersurface of L_2 vanishes identically. This is also obvious from the fact that this hypersurface being one dimensional $\Gamma^\alpha_{\beta\gamma}$ must be symmetric in β, γ. Carrying out the induced partial covariant differentiation of (2.12) and using the identity

$$\nabla_\beta T_i = B^k_\beta \nabla_k T_i$$

we get

$$\nabla_\beta T_\gamma = \frac{n-2}{n-1} [(\nabla_j T_k) B^j_\beta B^k_\gamma + (\nabla_\beta B^k_\gamma) T_k]. \qquad ...(2.13)$$

By virtue of the relation

$$\nabla_\beta B^k_\gamma = B^k_{\gamma\beta} - B^k_\varepsilon \Gamma^\varepsilon_{\gamma\beta} + \Gamma^k_{hj} B^h_\gamma B^j_\beta$$

and eqns. (1.2) and (2.7) we obtain

$$\nabla_{[\beta} B^k_{\gamma]} = 0. \qquad ...(2.14)$$

Equation (2.13) will therefore give

$$\nabla_{[\beta} T_{\gamma]} - \frac{n-2}{n-1} \nabla_{[j} T_{k]} B^j_\beta B^k_\gamma. \qquad ...(2.15)$$

The connection parameter Γ^i_{jk} (or $\Gamma^\alpha_{\beta\gamma}$) is said to be the connection of Enghis (1972) if and only if the torsion vector T_k (or T_γ) is irrotational.

Equation (2.15) proves the following.

Theorem 2.4 — For $n > 2$ the induced connection parameter $\Gamma^\alpha_{\beta\gamma}$ is a connection of Enghis (1972) provided that Γ^i_{jk} is such a connection.

The following two theorems are immediate consequences of eqns. (2.11) and (2.15).

Theorem 2.5 — If the vector field S_k is irrotational then S_γ is also irrotational.

Theorem 2.6 — If the vector field S_k is irrotational then the connection parameters Γ^i_{jk} and $\Gamma^\alpha_{\beta\gamma}$ are connections of Enghis (1972). Golab (1975) has proved that if a connection is Γ^{**}, it is a connection of Enghis (1972). This fact and Theorem 2.2 proves the following.

Theorem 2.7 — If Γ^{i}_{jk} is Γ^{**}, then both Γ^{i}_{jk} and $\Gamma^{\alpha}_{\beta\gamma}$ are connections of Enghis (1972).

3. Semi-symmetric and Quarter-symmetric Metric Connection Parameters in Finsler Spaces

Let the space L_n and L_{n-1} referred to in the last two sections be Finsler spaces denoted by F_n and F_{n-1} respectively. A semi-symmetric metric connection in F_n may be defined by [similar to the definition given by Imai (1972, 1973)],

$$\Gamma^{i}_{jk} = \Gamma^{*i}_{jk} + \delta^{i}_{j} S_k - g_{jk} S^i, \qquad ...(3.1)$$

where $g_{jk}(x, \dot{x})$ and Γ^{*i}_{jk} are the components of the metric tensor and Cartan's connection coefficients of F_n and

$$S^i = g^{ij} S_j.$$

A quarter-symmetric metric connection in F_n will be defined by

$$\Gamma^{i}_{jk} = \Gamma^{*i}_{jk} + t^{i}_{j} S_k - g_{jk} S^i. \qquad ...(3.2)$$

A simple calculation based on eqns. (2.4), (3.2) and the relation

$$\Gamma^{*\alpha}_{\beta\gamma} = B^{\alpha}_{i} (B^{i}_{\beta\gamma} + \Gamma^{*i}_{jk} B^{j}_{\beta} B^{k}_{\gamma}), \; B^{\alpha}_{i} = g^{\alpha\beta} g_{ij} B^{j}_{\beta}$$

yields

$$\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{*\alpha}_{\beta\gamma} + t^{\alpha}_{\beta} S_\gamma - g_{\beta\gamma} S^\alpha \qquad ...(3.3)$$

where $g_{\beta\gamma}$, $\Gamma^{*\alpha}_{\beta\gamma}$ are the components of the metric tensor and induced connection parameter in F_{n-1}; S_γ and t^{α}_{β} have been defined by (2.8) and (2.10); and

$$S^\alpha = g^{\alpha\beta} S_\beta.$$

Defining

$$p^{i}_{\alpha\beta} = \nabla_\beta B^{i}_{\alpha} = B^{i}_{\alpha\beta} - B^{i}_{\varepsilon} \Gamma^{\varepsilon}_{\alpha\beta} + \Gamma^{i}_{jk} B^{j}_{\alpha} B^{k}_{\beta} \qquad ...(3.4)$$

and simplifying with the help of relations (3.2) and (3.3), we get

$$p^{i}_{\alpha\beta} = I^{i}_{\alpha\beta} - B^{i}_{\varepsilon} (t^{\varepsilon}_{\alpha} S_\beta - g_{\alpha\beta} S^\varepsilon) + (t^{i}_{j} S_k - g_{jk} S^i) B^{j}_{\alpha} B^{k}_{\beta} \qquad ...(3.5)$$

where

$$I^i_{\alpha\beta} = B^i_{\alpha\beta} - B^i_\varepsilon \Gamma^{*\varepsilon}_{\alpha\beta} + \Gamma^{*i}_{jk} B^j_\alpha B^k_\beta .$$

In order to simplify the relation (3.5) we note that

$$S^i = S^\alpha B^i_\alpha + (N_j S^j) N^i, \; B^\alpha_j B^i_\alpha = \delta^i_j - N^i N_j \qquad ...(3{\cdot}6)$$

where the unit vector N^i is normal to the hypersurface. A simplification based on eqns. (3.4), (3.5) and (3.6) will yield

$$p^i_{\alpha\beta} = I^i_{\alpha\beta} - (N_j S^j) N^i g_{\alpha\beta} + N^i V_\alpha S_\beta \qquad ...(3.7)$$

where

$$V_\alpha = t^k_l N_k B^l_\alpha . \qquad ...(3.8)$$

Putting $\Omega_{\alpha\beta} = N_i I^i_{\alpha\beta}$ and $\bar{\Omega}_{\alpha\beta} = N_i p^i_{\alpha\beta}$, we get

$$\bar{\Omega}_{\alpha\beta} = \Omega_{\alpha\beta} - (N_j S^j) g_{\alpha\beta} + V_\alpha S_\beta . \qquad ...(3.9)$$

The tensor $\Omega_{\alpha\beta}$ has been called the second fundamental tensor of the hypersurface. The tensor $\bar{\Omega}_{\alpha\beta}$ which is a generalized form of $\Omega_{\alpha\beta}$ will be called '*K*-second fundamental tensor' of F_{n-1}.

If the connection Γ^i_{jk} is semi-symmetric, $V_\alpha = 0$. Therefore,

$$\bar{\Omega}_{\alpha\beta} = \Omega_{\alpha\beta} - (N_j S^j) g_{\alpha\beta} . \qquad ...(3.10)$$

Equations (3.8) and (3.9) lead to the following theorems.

Theorem 3.1 — If the vector field S_i is tangent to the hypersurface or $t^i_j = P^i Q_j$ is the outer product of two vectors P and Q of which either P is tangent or Q is normal to F_{n-1} then *K*-second fundamental tensor is equal to the second fundamental tensor of the hypersurface.

The following two theorems are direct consequences of eqn. (3.10).

Theorem 3.2 — The necessary and sufficient condition that *K*-second fundametal tensor arising from a semi-symmetric connection parameter be equal to the second fundamental tensor is that the vector S^i is tangent to the hypersurface.

Theorem 3.3 — A point of the hypersurface is umbilical if and only if $\bar{\Omega}_{\alpha\beta}$ is proportional to $g_{\alpha\beta}$.

It may be noted that an umbilical point is characterized by the relation

$$\Omega_{\alpha\beta} = \rho g_{\alpha\beta}$$

where ρ is a scalar.

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