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ON EINSTEIN-KAEHLERIAN CONHARMONIC RECURRENT SPACES

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In the present paper, the author has defined and studied Einstein-Kaehlerian conharmonic recurrent spaces and Einstein-Kaehlerian spaces with recurrent Bochner curvature tensor and several theorems have been established. The necessary and sufficient condition for an Einstein-Kaehlerian conharmonic recurrent space to be Kaehlerian recurrent has been investigated.

1. INTRODUCTION

Mathai (1969) and Walker (1950) have studied Kaehlerian recurrent spaces and Ruse's spaces of recurrent curvature respectively. Matsumoto (1969) has considered Kaehlerian spaces with parallel or vanishing Bochner curvature tensor. Further, Singh and Singh (1977) have studied and defined Kaehlerian conharmonic recurrent and Kaehlerian conharmonic symmetric spaces and several theorems have been investigated.

Here, we define Einstein-Kaehlerian conharmonic recurrent spaces and Einstein-Kaehlerian spaces with recurrent Bochner curvature tensor and establish several theorems. The necessary and sufficient condition for an Einstein-Kaehlerian conharmonic recurrent space to be Kaehlerian recurrent is investigated.

An $n(= 2m)$ dimensional Kaehlerian space is a Riemannian space which admits a tensor field F_i^h satisfying

$$F_j^h F_h^i = -\delta_j^i \qquad ...(1.1)$$

$$F_{ij} = -F_{ji}, \quad (F_{ij} = F_i^a g_{aj}) \qquad ...(1.2)$$

and

$$F_{i,j}^h = 0 \qquad ...(1.3)$$

where the comma (,) followed by an index denotes the operation of covariant differentiation with respect to the metric tensor g_{ij} of the Riemannian space. Let

$$R_{ijk}^h = \partial_i \left\{ {h \atop jk} \right\} - \partial_j \left\{ {h \atop ik} \right\} + \left\{ {h \atop il} \right\} \left\{ {l \atop jk} \right\} - \left\{ {h \atop jl} \right\} \left\{ {l \atop ik} \right\} \qquad ...(1.4)$$

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$R_{jk} = R^{i}_{ijk}$ and $R = R_{jk}g^{jk}$ be the Riemannian curvature tensor, the Ricci tensor and scalar curvature respectively.

Recently, Tachibana (1967) has defined the Bochner curvature tensor (with respect to a real coordinate system)

$$K^{h}_{ijk} = R^{h}_{ijk} + \frac{1}{n+4}(R_{ik}\delta^{h}_{j} - R_{jk}\delta^{h}_{i} + g_{ik}R^{h}_{j} - g_{jk}R^{h}_{i} + S_{ik}F^{h}_{j} - S_{jk}F^{h}_{i} + F_{ik}S^{h}_{j} - F_{jk}F^{h}_{i} + 2S_{ij}F^{h}_{k} + 2F_{ij}S^{h}_{k}) - \frac{R}{(n+2)(n+4)} \times (g_{ik}\delta^{h}_{j} - g_{jk}\delta^{h}_{i} + F_{ik}F^{h}_{j} - F_{jk}F^{h}_{i} + 2F_{ij}F^{h}_{k}) \quad ...(1.5)$$

where

$$S_{ij} = F^{a}_{i} R_{aj}.$$

The Kaehlerian conharmonic curvature tensor (1973) is given by

$$T^{h}_{ijk} = R^{h}_{ijk} + \frac{1}{n+4}(R_{ik}\delta^{h}_{j} - R_{jk}\delta^{h}_{i} + g_{ik}R^{h}_{j} - g_{jk}R^{h}_{i} + S_{ik}F^{h}_{j} - S_{jk}F^{h}_{i} + F_{ik}S^{h}_{j} - F_{jk}S^{h}_{i} + 2S_{ij}F^{h}_{k} + 2F_{ij}S^{h}_{k}). \quad ...(1.6)$$

Let us suppose that a Kaehlerian space is an Einstein one, then the Ricci tensor satisfies

$$R_{ij} = \frac{R}{n} g_{ij}, \; R_{,a} = 0 \quad ...(1.7)$$

from which we obtain

$$R_{ij,a} = 0, \; S_{ij,a} = 0 \text{ and } S_{ij} = \frac{R}{n} F_{ij}. \quad ...(1.8)$$

If a Kaehlerian space is an Einstein one, then the Bochner curvature tensor and the Kaehlerian conharmonic curvature tensor reduce to the forms :

$$U^{h}_{ijk} = R^{h}_{ijk} + \frac{R}{n(n+2)}(g_{ij}\delta^{h}_{j} - g_{jk}\delta^{h}_{i} + F_{ik}F^{h}_{j} - F_{jk}F^{h}_{i} + 2F_{ij}F^{h}_{k}) \quad ...(1.9)$$

and

$$E^h_{ijk} = R^h_{ijk} + \frac{2R}{n(n+4)} (g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k). \quad ...(1.10)$$

respectively.

In view of eqns. (1.9) and (1.10), we have

$$U^h_{ijk} = E^h_{ijk} - \frac{R}{(n+2)(n+4)} (g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k). \quad ...(1.11)$$

We shall use the following :

Definition (Lal and Singh 1971) — A Kaehler space is said to be recurrent if we have

$$R^h_{ijk,a} - \lambda_a R^h_{ijk} = 0 \quad ...(1.12)$$

for some non-zero recurrence vector λ_a and is called Ricci-recurrent if it satisfies the relation

$$R_{ij,a} - \lambda_a R_{ij} = 0. \quad ...(1.13)$$

Multiplying the above equation by g^{ij}, we get

$$R_{,a} - \lambda_a R = 0. \quad ...(1.14)$$

2. Einstein-Kaehlerian Conharmonic Recurrent Spaces

Definition 2.1 — A Kaehler space satisfying the relation

$$E^h_{ijk,a} - \lambda_a E^h_{ijk} = 0 \quad ...(2.1)$$

where λ_a is a non-zero recurrence vector, will be called an Einstein-Kaehlerian conharmonic recurrent space or briefly an E-K^* space.

Definition 2.2 — A Kaehler space satisfying the relation

$$U^h_{ijk,a} - \lambda_a U^h_{ijk} = 0 \quad ...(2.2)$$

where λ_a is a non-zero recurrence vector, will be called an Einstein-Kaehlerian space with recurrent Bochner curvature tensor.

We have the following :

Theorem 2.1 — A necessary and sufficient condition for an E-K^* space to be a Kaehlerian recurrent is that the scalar curvature be equal to zero.

PROOF : Suppose that an E-K^* space is Kaehlerian recurrent. Making use of eqns. (1.7), (1 8) and (1.10) in (2.1), we obtain

$$R^h_{ijk,a} = \lambda_a \Big[R^h_{ijk} + \frac{2R}{n(n+4)} (g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k) \Big]. \quad \ldots(2.3)$$

Since, an E-K^* space is Kaehlerian recurrent, eqn. (2.3) reduces to

$$\frac{2R}{n(n+4)} (g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k) = 0, \quad \ldots(2.4)$$

which gives $R = 0$.

Conversely, if an E-K^* space satisfies $R = 0$, then eqn. (2.3) reduces to

$$R^h_{ijk,a} - \lambda_a R^h_{ijk} = 0,$$

which shows that the space is Kaehlerian recurrent.

This completes the proof.

Similarly, in view of Theorem 2.1 and eqns. (1.7), (1.8) and (1.11), we can prove the following theorem :

Theorem 2.2 — A necessary and sufficient condition for an Einstein-Kaehlerian space with recurrent Bochner curvature tensor to be a Kaehlerian recurrent is that the scalar curvature be equal to zero.

Now, from (1.8) and the Bianchi identity

$$R^h_{ijk,a} + R^h_{jak,i} + R^h_{aik,j} = 0 \quad \ldots(2.5)$$

we have

$$R^l_{ijk,l} = 0.$$

Thus, contracting (2.3) with respect to h and a, we obtain

$$\lambda_l R^l_{ijk} + \frac{2R}{n(n+4)} (\lambda_j g_{ik} - \lambda_i g_{jk} + \lambda_l F_{ik}F^l_j - \lambda_l F_{jk}F^l_i + 2\lambda_l F_{ij}F^l_k) = 0. \quad \ldots(2.6)$$

Furthermore, substituting (2.3) in (2.5) and then transvecting with respect to λ^a, we get

$$\lambda^l\lambda_l R^h_{ijk} + \lambda^l\lambda_l \frac{2R}{n(n+4)} (g_{ik}\delta^h_j - g_{jk}\delta^h_i$$

$$+ F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k) + \lambda_i \left\{\lambda^l R^h_{jlk}\right.$$

$$+ \frac{2R}{n(n+4)} (\lambda^h g_{jk} - \lambda_k\delta^h_j + \lambda^a F_{jk}F^h_a - \lambda^a F_{ak}F^h_j$$

$$\left. + 2\lambda^a F_{ja}F^h_k)\right\} + \lambda_j \left\{\lambda^l R^h_{lik} + \frac{2R}{n(n+4)} (\lambda_k\delta^h_i\right.$$

$$\left. - \lambda^h g_{ik} + \lambda^a F_{ak}F^h_i - \lambda^a F_{ik}F^h_a + 2\lambda^a F_{ai}F^h_k)\right\} = 0 \qquad \ldots(2.7)$$

which in view of (2.6) gives

$$\lambda^l\lambda_l \left\{R^h_{ijk} + \frac{2R}{n(n+4)} (g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j\right.$$

$$- F_{jk}F^h_i + 2F_{ij}F^h_k) = 0. \qquad \ldots(2.8)$$

Since λ_a is a non-zero vector, we have the following :

Theorem 2.3 — An E-K^* space is the space of constant holomorphic sectional curvature (Yano 1965).

We have the following Lemmas from Walker (1950) and Yano and Bochner (1953).

Lemma 2.1 — The curvature tensor R_{hijk} satisfies the identity

$$R_{hijk,lm} - R_{hijk,ml} + R_{jklm,hi} - R_{jklm,ih} + R_{lmhi,jk} - R_{lmhi,kj} = 0, \qquad \ldots(2.9)$$

where $R_{hijk,l,m} \overset{def}{=} R_{hijk,lm}$.

Lemma 2.2 — If $a_{\alpha\beta}$, b_γ are quantities satisfying

$$a_{\alpha\beta} = a_{\beta\alpha} \text{ and } a_{\alpha\beta}b_\gamma + a_{\beta\gamma}b_\alpha + a_{\gamma\alpha}b_\beta = 0 \qquad \ldots(2.10)$$

for $\alpha, \beta, \gamma = 1, 2, \ldots, N$, then either all the $a_{\alpha\beta}$ are zero or all the b_γ are zero.

With the help of above Lemmas, we shall prove the following :

Theorem 2.4 — In an E-K^* space, either recurrence vector is gradient or the space is of constant holomorphic sectional curvature.

PROOF : Differentiating (2.3) covariantly and using eqns. (1.3), (1.7), (1.8) and (2.3), we obtain

$$R_{ijkh,ab} = (\lambda_{a,b} + \lambda_a\lambda_b)\, E_{ijkh} \qquad \ldots(2.11)$$

where

$$E_{ijkh} = R_{ijkh} + \frac{2R}{n(n+4)}(g_{ik}g_{hj} - g_{jk}g_{hi} + F_{ik}F_{jh} - F_{jk}F_{ih} + 2F_{ij}F_{kh}).$$

From (2.11) and the identity (2.9), we have

$$\lambda_{ab}E_{ijkh} + \lambda_{ij}E_{kha b} + \lambda_{kh}E_{abij} = 0, \qquad \ldots(2.12)$$

where $\lambda_{ab} \overset{def}{=} \lambda_{b,a} - \lambda_{a,b}.$

Equation (2.12) is of the form (2.10) since $E_{ijkh} = E_{khij}$. Thus, from Lemma 2.2, we have the Theorem 2.4.

Theorem 2.5 — An E-K^* space is the Einstein-Kaehlerian space with recurrent Bochner curvature tensor iff the scalar curvature $R = 0$.

PROOF : Suppose that an E-K^* space is the Einstein-Kaehlerian space with recurrent Bochner curvature.

Differentiating (1.11) covariantly with respect to x^a and using (1.7), we have

$$U^h_{ijk,a} = E^h_{ijk,a}. \qquad \ldots(2.13)$$

Multiplying eqn. (1.11) by λ_a and subtracting from (2.13), we get

$$U^h_{ijk,a} - \lambda_a U^h_{ijk} = E^h_{ijk,a} - \lambda_a E^h_{ijk} + \frac{\lambda_a R}{(n+2)(n+4)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i$$
$$+ F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k). \qquad \ldots(2.14)$$

Now, making use of the above supposition, eqn. (2.14) reduces to

$$\frac{\lambda_a R}{(n+2)(n+4)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k) = 0, \ldots(2.15)$$

which implies $R = 0$.

Conversely let us suppose that in an E-K^* space, the scalar curvature $R = 0$. Hence, eqn. (2.14) reduces to

$$U^h_{ijk,a} - \lambda_a U^h_{ijk} = 0 \qquad \ldots(2.16)$$

which shows that the space is an Einstein-Kaehlerian space with recurrent Bochner curvature tensor.

This completes the proof.

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