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ON REDUCING SUBSPACES OF COMPLETELY NON-NORMAL OPERATORS

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In the present work we characterize the reducing subspaces of a completely non-normal operator T in a complex Hilbert Space H in terms of the range of the self-commutator $T^*T - TT^*$.

Let T be any bounded linear operator on a complex Hilbert space H. A (closed) subspace M of H reduces T if M is invariant under both T and T^*. An operator T on H is called completely non-normal (abnormal) if there is no non-trivial subspace M of H which reduces T and such that the restriction of T to M is normal. Let $B(H)$ stand for the algebra of all bounded linear operators on H. Let $T \mid M$ stand for the restriction of an operator T to a subspace M of H.

In the present paper we characterize the reducing subspaces of a completely non-normal operator T in $B(H)$ in terms of the range of the self commutator $T^*T - TT^*$. Our work has been motivated by the work of Guyker (1974) for a similar problem for contraction in $B(H)$ with no isometric part.

If E is a subset of H, then $\vee E$ will denote the closed span of E. For subspaces M and N of H such that $N \subset M$, $M \ominus N$ will denote the orthogonal complement of N in M. We now state and prove our main result and give some corollaries.

Theorem 1 — Let $T \in B(H)$ be completely non-normal. Let K be the closure of the range of the self-commutator $T^*T - TT^*$. A subspace M of H reduces T if and only if $M = \vee \{(T^{*m}T^n - T^nT^{*m}) f : f \in S,\ m, n$ are nonnegative integers$\}$ for some unique subspace S of K which is invariant under $(T^*T - TT^*)(T^{*m}T^n - T^nT^{*m})$ for every m, n taking nonnegative integer values. In this case

$$H \ominus M = V\{(T^{*m}T^n - T^nT^{*m}) f : f \in K \ominus S,\ m, n \text{ are nonnegative integers}\}.$$

PROOF : Suppose M reduces T and let S be the closure of $(T^*T - TT^*) M$ in H. Clearly $P = \{(T^{*m}T^n - T^nT^{*m}) f : f \in S,\ m, n$ are nonnegative integers$\} \subset M$, and if g is in $M \ominus P$, then

$$\langle g, (T^{*m}T^n - T^nT^{*m})(T^*T - TT^*) g \rangle = 0$$

for m, n taking nonnegative integer values, from which it follows that

$$\| (T^*T - TT^*) g \| = 0$$

which implies that

$$(T^*T - TT^*) g = 0. \qquad \ldots(*)$$

But since T is completely non-normal, hence $g = 0$. Therefore,

$$M = \bigvee \{(T^{*m}T^n - T^nT^{*m}) f : f \in S,\ m, n \text{ are nonnegative integers}\}.$$

Since M reduces T, we have that $K \ominus S$ is the closure in H of $(T^*T - TT^*) (H \ominus M)$ and therefore,

$$H \ominus M = \bigvee \{(T^{*m}T^n - T^nT^{*m}) f : f \in K \ominus S,\ m, n \text{ are nonnegative integers}\}.$$

Conversely suppose that,

$$M = \bigvee \{(T^{*m}T^n - T^nT^{*m}) f : f \in S',\ m, n \text{ are nonnegative integers}\}$$

where S' is a subspace of K which is invariant under

$$(T^*T - TT^*) (T^{*m}T^n - T^nT^{*m}) \ (m, n \text{ are nonnegative integers}).$$

Let $$N = \{g \in H : (T^*T - TT^*) (T^{*m}T^n - T^nT^{*m}) g \in S',\ m, n \text{ are nonnegative integers}\}$$

Clearly $M \subset N$. Let $g \in N$. Since,

$$(T^*T - TT^*) (T^{*m}T^n - T^nT^{*m}) g \in S', \text{ we get}$$

$$(T^{*m}T^n - T^nT^{*m}) (T^*T - TT^*) (T^{*m}T^n - T^nT^{*m}) g \text{ is in } M.$$

It now follows by the same argument as in deriving (*) that $N = M$ and hence M reduces T.

Let S be the closure of $(T^*T - TT^*) M$ in H. Clearly $S \subset S'$. Since $(T^*T - TT^*) (T^{*m}T^n - T^nT^{*m}) f \in S'$, for every $f \in S'$ and m, n are nonnegative integers.

As above

$$H \ominus M = \bigvee \{(T^{*m}T^n - T^nT^{*m}) f : f \in K \ominus S,\ m, n \text{ are nonnegative integers}\}$$

It follows that $S' \ominus S$ is both contained in M and $H \ominus M$ and cosequently $S' = S$ thus uniqueness of S is established, and that completes the proof of the theorem.

The following result has been derived by Morrel (1973) with a different technique. We state the result since it also follows as a corollary to our result.

Corollary 1 — If $T \in B(H)$ is a completely non-normal operator and $T^*T - TT^*$ is of one dimensional range, then T is irreducible.

PROOF : In the above theorem let K be one dimensional. Since the only subspace S of K are $\{0\}$ and K itself, therefore it follows that the only subspaces of H which reduce T are $\{0\}$ and H itself. Another result of Morrel (1973) is as follows:

Corollary 2 — Let $T \in B(H)$. Then the largest subspace H_0 of H reducing T such that $T \mid H_0$ is normal is given by

$$H_0 = \bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \text{Ker} \, (T^{*m}T^n - T^nT^{*m})$$

PROOF : We omit the proof because it is easy.

Remarks : It has been brought to the notice of the author by the referee of an announcement by S. K. Khasbardar and N. K. Thakare (*Notices Am. math. Soc.*, **76** (1976), T-B 116 p. A440) which is due to appear in full in *Math. Student* [**43**, (1976)] that the results of Guyker (1974) are still valid for any operator with no isometric part.

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