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GALAXY FORMATION IN EDDINGTON-LEMAITRE UNIVERSE

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In this paper, it is shown that out of the three non-elliptic solutions of the generalised Friedmann equation, one corresponds to the Eddington-Lemaitre universe. This model is obtainable by the perturbation of Einstein static model as a sequel of the occurrence of gravitational instability. It is shown that the density fluctuations grow exponentially in this model. This provides a justification of the Eddington-Lemaitre model as a suitable model for the formation of protogalaxies in the universe.

INTRODUCTION

Bondi (1961) and Rindler (1969) have pointed out that the general soultion of the Friedmann equation is quite intricate as it involves elliptic functions; however, we have confined our investigations in this paper to non-elliptic solutions. In the first part of this paper, we have found out three non-elliptic solutions of the Friedmann equation out of which one solution corresponds to Eddington-Lemaitre universe. As is well known, Einstein static model is unstable against random density fluctuations and eventually transforms to Eddington-Lemaitre universe. In the second part of the paper, we have shown by using tetrad technique (Johri 1969) that the density fluctuations grow exponentially in the Eddington-Lemaitre universe and that the statistical density fluctuations might eventually grow into galactic clusters within suitable time scale. Simon (1970) obtained similar results by using a classical technique due to Landau and Lifshitz.

PART A

Investigating the dust filled cosmological models of the expanding universe, Friedmann (1922) obtained a differential equation which was later generalized by adjunction of Λ term. It covers all general relativistic dust universes that are isotropic and homogeneous.

The generalized Friedmann equation is

$$\dot{R}^2 = \frac{c}{R} - k + \tfrac{1}{3}\Lambda R^2 \qquad \ldots(1)$$

where the constant c is related to the mass of universe, k is spatial curvature and R the scale factor.

Bondi (1961) has given a classification of the solutions of (1) by variation of Λ and k. The nature of the most intricate case, corresponding to $\Lambda \neq 0$, $k \neq 0$, has been discussed by him by qualitative integration. However, Pathak (1975) has obtained the exact solution of (1) in terms of non-elliptic functions. We discuss here the cosmological importance of this solution and show that the density fluctuations can grow exponentially with time in such a cosmological model.

As shown by Pathak (1975) eqn. (1) is exactly integrable in terms of non-elliptic functions under three cases given below :

$$\text{(i)} \quad c = 0, \quad \text{(ii)} \quad c = +\frac{2k^{3/2}}{3\Lambda^{1/2}}, \quad \text{(iii)} \quad c = -\frac{2k^{3/2}}{3\Lambda^{1/2}}.$$

Case I : $c = 0$

This is the well known case in which the Friedmann equation reduces to the form

$$\left(\frac{dR}{dt}\right)^2 = \tfrac{1}{3}\Lambda R^2 - k \qquad \ldots(2)$$

and yields the following solutions for different values of curvature :

$$k = 0, \; R \sim \exp\,[t(\Lambda/3)^{1/2}],$$

$$(k = +1) > 0, \; R = \sqrt{\frac{3}{\Lambda}}\cosh\left[t\sqrt{\frac{\Lambda}{3}} + A'\right],$$

$$(k = -1) < 0, \; R = \sqrt{\frac{3}{\Lambda}}\sinh\left[t\sqrt{\frac{\Lambda}{3}} + A''\right].$$

Case II : $c = +\dfrac{2k^{3/2}}{3\Lambda^{1/2}} > 0$

In this case we have

$$\sinh^{-1}\frac{u}{\sqrt{2}}\left(\frac{\Lambda}{k}\right)^{1/4} + \int\frac{dy}{y^2 - 3} = \sqrt{\frac{\Lambda}{12}}\,t + B' \qquad \ldots(3)$$

where $u = \sqrt{R}$, $y^2 = 1 + \dfrac{2k^{1/2}}{\Lambda^{1/2}R}$.

It is found that $y^2 > 3$ corresponds to limit

$$R < \frac{k^{1/2}}{\Lambda^{1/2}}.$$

The corresponding solution is

$$\sinh^{-1}\left(\frac{R^2\Lambda}{4k}\right)^{1/4} + \frac{1}{\sqrt{12}}\log\frac{y - \sqrt{3}}{y + \sqrt{3}} = \sqrt{\frac{\Lambda}{12}}\,t + B'. \qquad \ldots(4)$$

For small values of R, y is large; therefore the second term on left-hand side of (4) vanishes and the remaining terms indicate that $B' = 0$ assuming the occurrence of a point singularity in Friedmann models. It is evident from eqn. (1.4) that $R \to \frac{k^{1/2}}{\Lambda^{1/2}}$ asymptotically from below.

In the case $y^2 < 3$ i.e. $R > \frac{k^{1/2}}{\Lambda^{1/2}}$, eqn. (1.3) yields

$$\sinh^{-1}\left(\frac{R^2\Lambda}{4k}\right)^{1/4} - \frac{1}{\sqrt{12}} \log \frac{\sqrt{3} + y}{\sqrt{3} - y} = \sqrt{\frac{\Lambda}{12}}\, t + B'. \qquad \text{...(5)}$$

For large values of R, y is sufficiently small as such the second term on the left-hand side of (1.5) vanishes and we have

$$R \sim \exp\,[t(\tfrac{1}{3}\Lambda)^{1/2}]. \qquad \text{...(6)}$$

Further as R approaches $R_c \left(= \frac{k^{1/2}}{\Lambda^{1/2}}\right)$ from above, the second term of left-hand side of (5) tends to $-\infty$, showing thereby that $R \to R_c$ asymptotically from above as $t \to -\infty$.

The above mentioned characteristics of the solution testify that the exact non-elliptic solution obtained by us is nothing but Eddington-Lemaitre universe discussed by Bondi (1961) qualitatively.

Case III : $c = -\frac{2k^{3/2}}{3\Lambda^{1/2}} < 0$

The exact non-elliptic solution of the Friedmann equation turns out to be

$$\cosh^{-1}\left(\frac{R^2\Lambda}{4k}\right)^{1/4} - \frac{1}{\sqrt{12}} \log \frac{\sqrt{3} + y}{\sqrt{3} - y} = \sqrt{\frac{\Lambda}{12}}\, t + B''. \qquad \text{...(7)}$$

This solution is physically untanable as $c < 0$ corresponds to negative density of matter.

Part B

Now we shall show that the Einstein static model is gravitationally unstable and a slight perturbation leads to Eddington-Lemaitre model in which the density perturbations are found to grow exponentially.

Let us take the Einstein model in the isotropic form

$$ds^2 = \frac{1}{\left(1 + \frac{\Upsilon^2}{4R^2}\right)^2} \{(dx^1)^2 + (dx^2)^2 + (dx^3)^2\} - dt^2. \qquad \text{...(8)}$$

Density and pressure for this model (Tolmann 1966) are given by

$$\left.\begin{aligned} 8\pi p &= -\frac{1}{R^2} + \Lambda \\ 8\pi\rho &= \frac{3}{R^2} - \Lambda. \end{aligned}\right\} \quad ...(9)$$

Thus if we assume the fluid to be composed of incoherent matter then

$$\left.\begin{aligned} \Lambda &= +\frac{1}{R^2} \\ \rho &= +\frac{1}{4\pi R^2} = \frac{\Lambda}{4\pi} = C_1 \text{ (a constant).} \end{aligned}\right\} \quad ...(10)$$

Now we consider the Einstein model as the background model for perturbation and fix up a system of orthonormal tetrad to this model. In the neighbourhood of $r = 0$, tetrad components are

$$\left.\begin{aligned} & e_1^1 = e_2^2 = e_3^3 = 1 \\ \text{and} \quad & e_i^j = 0 \; (i \neq j). \end{aligned}\right\} \quad ...(11)$$

The new set of tetrad fixed to the perturbed model is of the form

$$\left.\begin{aligned} e_1^1 &= 1 + \epsilon_{11}\,[x^0, x^\nu] \\ e_2^2 &= 1 + \epsilon_{22}\,[x^0, x^\nu] \\ e_3^3 &= 1 + \epsilon_{33}\,[x^0, x^\nu] \\ e_2^1 &= \pi_{12}\,[x^0, x^\nu] \\ e_3^1 &= \pi_{13}\,[x^0, x^\nu] \\ e_3^0 &= y e_3^3 \end{aligned}\right\} \quad ...(12)$$

so that

$$\left.\begin{aligned} \delta e_1^1 &= \epsilon_{11}, \; -\delta\theta_1 = \partial_0(\epsilon_{11}) \\ \delta e_2^2 &= \epsilon_{22}, \; -\delta\theta_2 = \partial_0(\epsilon_{22}) \\ \delta e_3^3 &= \epsilon_{33}, \; -\delta\theta_3 = \partial_0(\epsilon_{33}). \end{aligned}\right\} \quad ...(13)$$

Since the perturbed model is Eddington-Lemaitre model, so the tetrad perturbations in this particular case have the value

$$\delta e_1^1 = \delta e_2^2 = \delta e_3^3 \text{ and } \delta e_i^j = 0 \quad (i \neq j). \quad ...(14)$$

The growth of density fluctuations is given by perturbation of Raychaudhuri equation (Johri 1972). In the present case it reduces to the form

$$D^2(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) = \tfrac{1}{2}\,\frac{\delta\rho}{\rho}\cdot\rho\,. \qquad \ldots(15)$$

Denoting the contrast density by K, we get

$$DK = -\,(\delta\theta_1 + \delta\theta_2 + \delta\theta_3) = D(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) \qquad \ldots(16)$$

where $D \equiv \partial_0$,

as such eqn. (1.15) becomes

$$D^2K = \tfrac{1}{2}\,K\rho = \tfrac{1}{2}\,KC_1 \qquad \ldots(17)$$

which on integration gives

$$K = B_2 \exp\,(t\,\sqrt{C_1/2}) + B_3 \exp\,(-t\,\sqrt{C_1/2}) \qquad \ldots(18)$$

where C_1 is given by (10).

To discuss the growth of density fluctuations, let us consider the initial perturbation to be of the order of statistical fluctuation in an ideal gas; we have

$$K_i = N^{-1/2}$$

for collection of N molecules. For a normal galaxy $N \simeq 10^{68}$ as such $K_i \simeq 10^{-34}$ and the contrast density at the present epoch is $K_0 \simeq 10$. Thus, if an initial perturbation of the order of 10^{-34} in an ideal gas is magnified to the order of contrast density of galactic super clusters (Sciama 1968) at the present epoch, then we have

$$\frac{K_0}{K_i} = \frac{\exp\,((\sqrt{C_1/2})\,t_0)}{\exp\,((\sqrt{C_1/2})\,t_i)} \qquad \ldots(19)$$

which gives

$$t_0 - t_i \simeq 3.5 \times 10^9 \text{ years.}$$

The density perturbations grow exponentially in this model and the statistical fluctuations in the primordial state might grow to the size of observable galactic clusters within a span of 3.5×10^9 years, thus leading to formation of protogalaxies. It is worth mentioning in this context that Barrow and Matzner (1977) have favoured the idea of growth of density fluctuations right from the primordial stage.

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