Indian J. pure appl. Math., **10**(4) : 502-504, April 1979

REGULAR BOREL MEASURES AND COMPOSITION OPERATORS ON $L^p(\mu)$

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(*Received* 17 *August* 1978)

Let μ be the Lebesgue measure on the Borel subsets of the real line R and let ϕ be a measurable real valued function on R. Then a characterization of composition operator C_ϕ in terms of the measure $\mu\phi^{-1}$ is given in this paper.

1. PRELIMINARIES

Let (X, s, λ) be a measure space and let ϕ be a measurable non-singular ($\lambda\phi^{-1}(E) = \lambda(\phi^{-1}(E)) = 0$ whenever $\lambda(E) = 0$) transformation from X into itself. Let C_ϕ be a linear transformation from $L^p(X, s, \lambda)$ (briefly written as $L^p(\lambda)$) into the space of all complex valued functions on X defined by

$$C_\phi f = f \circ \phi \text{ for all } f \in L^p(\lambda).$$

If the range of C_ϕ is contained in $L^p(\lambda)$ and C_ϕ is a bounded linear transformation, then we call it a composition operator on $L^p(\lambda)$ induced by ϕ.

Every measurable transformation ϕ on X into itself induces a measure on (X, s) defined as $\lambda\phi^{-1}(E) = \lambda(\phi^{-1}(E))$ for all $E \in s$. It has been shown by Singh (1974, 1975) and Ridge (1973) that if C_ϕ is a composition operator on $L^p(\lambda)$, then the measure $\lambda\phi^{-1}$ is absolutely continuous with respect to the measure λ and the Radon-Nikodym derivative g_0 of the measure $\lambda\phi^{-1}$ with respect to the measure λ is essentially bounded, and $\| C_\phi \|^p = \| g_0 \|_\infty$.

In this note we are interested in the case where $X = R$, the set of real numbers, $s = B$, σ-algebra of Borel subsets of R and $\lambda = \mu$, the Lebesgue measure on the Borel subsets of R.

2. REGULARITY OF THE MEASURE $\mu\phi^{-1}$ AND COMPOSITION OPERATOR

Definition — A Borel measure λ on (R, B) is said to be regular if every Borel set is regular with respect to λ. [For definition of regular set see Rudin (1966)].

If $\phi : R \to R$ is a Borel measurable function, then $\mu\phi^{-1}$ is a Borel measure on R. We shall note the following result.

Proposition — Let $\phi : R \to R$ be a Borel measurable function. Then C_ϕ is bounded implies that $\mu\phi^{-1}$ is a regular Borel measure on R.

PROOF : In the light of theorem 2.18 of Rudin (1966) it is enough to show that $\mu\phi^{-1}(F) < \infty$ for every compact subset F of R. Since C_ϕ is bounded,

$$\| C_\phi g \| \leqslant \| C_\phi \| \| g \| \text{ for every } g \in L^p(\mu).$$

Let $g = X_F$, the characteristic function of F. Then from the above inequality it follows that

$$(\int | X_{\phi-1(F)} |^p d\mu)^{1/p} \leqslant \| C_\phi \| (\int | X_F |^p d\mu)^{1/p}$$

i.e. $$(\mu\phi^{-1}(F))^{1/p} \leqslant \| C_\phi \| (\mu(F))^{1/p}$$

i.e. $$\mu\phi^{-1}(F) \leqslant \| C_\phi \|^p \mu(F) < \infty.$$

This shows that $\mu\phi^{-1}$ is regular.

The converse of the above proposition is not always true. We shall cite the following example.

Example — Let $\phi : R \to R$ be defined as

$$\phi(x) = \begin{cases} x & \text{if } x \in (-\infty, 0] \cup [1, \infty) \\ x^2 & \text{if } x \in (0, 1). \end{cases}$$

Then clearly $\mu\phi^{-1}$ is regular, but C_ϕ is not bounded as proved by Singh (1976).

Given a regular Borel measure λ on R, one can define the function f_λ on R by

$$f_\lambda(x) = \begin{cases} \lambda\,[0, x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\lambda\,[x, 0) & \text{if } x < 0. \end{cases}$$

From Hewitt and Stromberg (1965) f_λ is a non-decreasing real valued left continuous function on R. Hence f_λ induces Lebesgue Stieltjes measure on R and it follows from Theorem 19.48 of Hewitt an Stromberg (1965) that

$$d\lambda = df_\lambda.$$

Let $\phi : R \to R$ be any Borel measurable function and let $\lambda = \mu\phi^{-1}$. Then we get $f_{\mu\phi-1}$ and for the sake of convenience we shall denote it merely by f.

Now we shall prove the main theorem of this paper.

Theorem — Let ϕ be a real valued Borel measurable function on R. Then C_ϕ is a compositon operator on $L^p(\mu)$ if and only if f is absolutely continuous and f', the derivative of f, is essentially bounded.

PROOF : Suppose C_ϕ is a bounded operator on $L^p(\mu)$. Then by Singh (1976) the measure $\mu\phi^{-1}$ is absolutely continuous with respect to μ. By Theorem 19.53 of Hewitt and Stromberg (1965) it follows that f is absolutely continuous and

$$\frac{d\,\mu\phi^{-1}}{d\,\mu} = \frac{df}{d\,\mu} = f'$$

where $\frac{d\,\mu\phi^{-1}}{d\,\mu}$ is the Radon-Nikodym derivative of the measure $\mu\phi^{-1}$ with respect to the measure μ. Hence

$$\| f' \|_\infty = \left\| \frac{d\,\mu\phi^{-1}}{d\,\mu} \right\|_\infty = \| C_\phi \|^p < \infty.$$

This proves the 'only if' part of the theorem.

To prove the 'if' part let us assume that f is absolutely continuous and $\| f' \|_\infty < \infty$. Since f is absolutely continuous function, it follows from Theorem 19.53 of Hewitt and Stromberg (1965) that the measure $\mu\phi^{-1}$ is absolutely continuous with respect to the measure μ. By Theorem 19.61 of Hewitt and Stromberg (1965) we obtain,

$$\mu\phi^{-1}(E) \leqslant \int_E f' d\,\mu$$

for every Borel set E. Thus

$$\mu\phi^{-1}(E) \leqslant \| f' \|_\infty \, \mu(E).$$

Hence C_ϕ is bounded. With this the proof of the theorem is completed.

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