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A NOTE ON SUBNORMAL OPERATORS WITH FINITE RANK SELF-COMMUTATOR

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A representation theorem for a subnormal operator with a finite rank self-commutator is derived. This generalizes an earlier result by Clancey (1971).

Clancey (1971) proved that if A is a completely subnormal operator on a Hilbert space H and the self-commutator of A is of rank one, i.e., $A^*A - AA^* = (,b)b$ for some b in H, then A is unitarily equivalent to $\alpha U + \beta$ for some scalers α and β, where U is the unilateral shift. In the present paper we try to generalize this result for completely subnormal operators with finite rank self-commutator. In section 1 we give some preliminaries. We also give some definitions and known results. In section 2 we give our main result.

§1. A bounded operator A on a Hilbert space H is called subnormal in case A is the restriction of a normal operator N acting on a superspace R containing H where H is an invariant subspace of N. A bounded operator T on a Hilbert space H is called hyponormal in case it's self-adjoint self commutator $T^*T - TT^* = D$ is positive semi-definite ($D \geqslant 0$). If an operator A is subnormal then A is hyponormal (see Halmos 1967).

A subnormal (hyponormal) operator A is called completely subnormal (hyponormal) in case there are no non-trivial subspaces reducing A on which A is a normal operator. Clancey (1971) characterizes some subnormal singular integral operators mentioned in the following paragraph.

We specialize H as follows. Let $H = L^2(E)$, where E is a bounded measurable subset of the real line R.

Let T_b be the operator defined for f in $L^2(E)$ and b fixed in $L^\infty(E)$ by

$$T_b f(x) = x f(x) + \pi^{-1} b(x) \int_E \overline{b(t)}\, f(t)\, (x - t)^{-1}\, dt \text{ a.e.} \qquad \ldots(1)$$

(The singular integral is interpreted as Cauchy principle value, i.e., $\int = \lim\limits_{\varepsilon \to 0} \int\limits_{|x-t| \geqslant \varepsilon}$).

Operators T_b are hyponormal. In fact, $T_b^* T_b - T_b T_b^* = \dfrac{2}{\pi} (,b)\, b$.

Hyponormal subnormal singular integral operators of special kind have been characterized by Clancey (1971). This characterization is an application of Proposition (Clancey 1971) which we have stated in the beginning of the present paper. We reproduce the statement and the proof of the following result of Clancey (see also Morrel 1973) which we use in the sequel.

Theorem 1 (Clancey 1971) — Suppose A is completely subnormal on H, and let $D = A^*A - AA^*$, then the smallest subspace invariant under A and containing R_D (the range of D) is H.

PROOF : Assume that $M = \text{Span}\,(A^n R_D)_{n>0} \neq H$. The subspace $M^\perp$ is clearly A^* invariant. Moreover, $M^\perp \subseteq R_D^\perp$ so that for g in $M^\perp$ and f in H one has $(Df, Ag) = 0$. This follows since $R_D^\perp$ = Kernel (D) is A invariant. Suppose now that for $n \leqslant k$ and all g in $M^\perp$ one has established $(A^n Df, Ag) = 0$ for f in H. Then for f in H and g in $M^\perp$, $(A^{k+1}Df, Ag) = (A^k Df, A^*Ag)$. Using the fact that $Dg = 0$, we find $(A^{k+1}Df, Ag) = (A^k Df, AA^*g)$, which is zero by induction since A^*g is in $M^\perp$. We have just established that $M^\perp$ is A invariant so $M^\perp$ reduces A, and $A \mid M^\perp$ is a normal operator. Since A is completely subnormal it follows that $M^\perp$ is $\{0\}$, and the proof is complete.

§2. We start this section with the definition of the unilateral shift. The unilateral shift is the operator U defined on a complete orthonormal basis $\{e_n\}_{n\geqslant 0}$ of a Hilbert space H by $Ue_n = e_{n+1}$ for $n \geqslant 0$. An operator A is said to be of finite rank if it can be represented as $A = \sum_{i=1}^{n} \theta_i \otimes \theta_i = \sum_{i=1}^{n} (\,,\theta_i)\,\theta_i$ (see Kato 1966, p. 561) for $\{\theta_i\}_{i=1}^{n}$ belonging to H. In what follows $\langle\,,\,\rangle$ will stand for the inner product induced in a direct sum Hilbert space by the ordinary inner product $(\,,)$ in component Hilbert spaces. The following theorem gives our main result.

Theorem 2 — If A is completely subnormal on H and the self-commutator $A^*A - AA^*$ is of finite rank, i.e., $A^*A - AA^* = \sum_{i=1}^{n} \theta_i \otimes \theta_i$ for $\theta_i \in H (i = 1, 2, \ldots, n)$, then A is unitarily equivalent to $\alpha S + \beta$ for some scalars α and β, and S is an n-fold copy of the unilateral shift.

PROOF : Since R_D in this case is the span of the vectors $\{\theta_i\}_{i=1}^{n}$, due to the observation made in Theorem 1 we find that if f is in H and f is perpendicular to θ_i for any i, then Af is perpendicular to θ_i. It follows that $(A^*\theta_i, f) = (\theta_i, Af) = 0$. Therefore for some complex number z_i, we have the following relation :

$$A^*\theta_i = \bar{z}_i\theta_i. \qquad \ldots(2)$$

Now set $C_i = \| \theta_i \|^{-1} (A - z_i I)$ and $c_i = \| \theta_i \|^{-1} \theta_i$. Let $E = \bigoplus_{i=1}^{n} C_i$. Let N be the subspace spaned by $\{c_i\}_{i=1}^{n}$. Then it can be seen that $\langle Ey, y \rangle = 0$ and $\| Ey \| = 1$ for $y \in N$ since $E^* y = 0$ for $y \in N$ from (2). Also by induction it follows that $\langle E^n y, E^k y \rangle = 0$ where $n \neq k$ and $\| E^n y \| = 1$ for all n and k. From Theorem 1 it also follows that $c_1, c_2, \ldots, c_n$ are the cyclic vectors for E hence E is unitarily equivalent to an n-fold copy of the unilateral shift (see Halmos 1967). Next it is easy to recover A from the above representation of E and our result follows.

The following representation of $T_{\hat{b}}$ in terms of shifts have been derived by Clancey (1971). If $E = (-1, 1)$ and $\hat{b} = (1 - x^2)^{1/4}$ then the operator $T_{\hat{b}}$ shifts the Chebyshev functions of the second kind. To be more precise, define the Chebyshev polynomials of the second kind by $U_0 = 1$, $U_1 = x$, and the recurrence relation, $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$. Call $u_n(x) = (2/\pi)\, U_n(x)\, (1 - x^2)^{1/4}$ the nth Chebyshev function of the second kind, the set $u_n(x)$, $n \geqslant 0$ forms a complete orthonormal basis in $L^2(-1, 1)$ and $T_{\hat{b}}\, u_n(x) = u_{n+1}$ for $n \geqslant 0$.

Remarks : In case A is a subnormal operator with a finite rank self-commutator the representation of A becomes a singular integral operator with matrix valued symbols. Our generalization gives a scope of getting results analogus to that of Clancey as discussed in the previous para, which needs some elaborate calculations. We propose to give that in a future work.

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