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A NOTE ON A THEOREM OF RAY AND SINGH

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A valid proof of a (corrected) version of Theorem 1 of Ray and Singh (1978) is presented in this note.

Ray and Singh (1978) have stated the following :

Theorem 1 — Let K be a nonempty, bounded, closed and convex subset of a reflexive Banach space X and let K have normal structure. If $T : K \to K$ is continuous and satisfies

$$\| Tx - Ty \| \leqslant \frac{\| x - Tx \| \, \| x - Ty \| + \| y - Ty \| \, \| y - Tx \|}{\| x - Ty \| + \| y - Tx \|} \qquad \ldots(*)$$

for all x, y in K, $x \neq y$, then T has a unique fixed point in K.

Remark 1 : If for a pair of distinct points $x, y \in K$, $\| x - Ty \| + \| y - Tx \| = 0$, then the right-hand side expression in (*) is not defined. Hence some additional condition is required whenever $y = Tx (x = Ty)$ in order to arrive at the conclusion of the lemma of Ray and Singh (1978) (see Theorem A below).

Remark 2 : In the proof of Theorem 1 of Ray and Singh (1978) namely case 2—T is implicitly assumed to be linear. This is not one of the hypotheses mentioned in the statement reproduced above. Hence the proof is not valid when T is nonlinear.

The purpose of this note is to give a restatement of Theorem 1 of Ray and Singh (1978) and a valid proof of the same.

Theorem A — Let K be a nonempty, bounded, closed and convex subset of a reflexive Banach space and let K have normal structure. Let T, a continuous selfmap on K, satisfy for distinct x, y in K

$$\| Tx - Ty \| \leqslant \frac{\| x - Tx \| \, \| x - Ty \| + \| y - Ty \| \, \| y - Tx \|}{\| x - Ty \| + \| y - Tx \|}$$

$$\text{if } \| x - Ty \| + \| y - Tx \| > 0,$$

$$= 0 \text{ otherwise.}$$

Then T has a unique fixed point in K.

Remark 3 : Suppose there exists an x in K such that $x = T^2x$. Then $Tx = x$ in view of $\| x - Ty \| + \| y - Tx \| = 0$ $(y = Tx)$ and the condition on the map. Thus, if T^2 has a fixed point, then T has a fixed point. We may add that the hypothesis that "T^2 has at most one fixed point" would also lead to the above conclusion.

PROOF : In view of Remark 3, we need only consider the case $T^2x \neq x$ for any x in K. In this case for every x in K and every n, $\| T^{n-1}x - T^{n+1}x \| > 0$. (Here we note that in the proof of the Lemma of Ray and Singh (1978) the possibility that the factor $\| T^{n-1}x - T^{n+1}x \|$ may also become zero has been overlooked). Hence the cancellation of the factor $\| T^{n-1}x - T^{n+1}x \|$ in the computation of an estimate for $\| T^n x - T^{n+1}x \|$ [in the Lemma of Ray and Singh (1978)] is justified and the conclusion of the lemma holds.

Let K_1, H, y and r be as in the proof of Theorem 1 of Ray and Singh (1978). Since $y \in H$, H is nonempty. Also $H \subset K_1$ and K_1 bounded imply H is bounded. Moreover, if $\{x_n\}$ is a sequence of points in H converging to x in X, we have $x \in K_1$, since $H \subset K_1$ and K_1 is closed. Also $\| x_n - Tx_n \| \leqslant r$ leads to $\| x - Tx \| \leqslant r$. Thus $x \in H$, that is, H is closed. Again, $x \in H$ and $D(O(Tx)) \leqslant D(O(x))$ together with K_1 invariant under T yield $Tx \in H$. Hence H is a nonempty, bounded, closed set which is invariant under T.

Consider the set G, that is $\overline{co}(TH)$. Then G is evidently nonempty, bounded, closed and convex. The fact that G is a proper subset of K_1 follows as in Ray and Singh (1978). We now claim that $TG \subset G$.

First we show that $co(TH) \subset H$. Let $g = \sum_{i=1}^{n} \lambda_i Th_i$ $(\sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geqslant 0, h_i \in H \, \forall i)$ be an arbitrary point in $co(TH)$. We may assume that all λ_i's are positive and $n > 1$. If some Th_i's are equal, the number of terms in the convex combination for g with all Th_i's distinct reduces to, say, m. If $m = 1$ clearly $g \in TH \subset H$. Hence we need consider the case where λ_i's are positive, $n > 1$ and Th_i's are distinct. Now

$$\| g - Tg \| = \left\| \sum_{i=1}^{n} \lambda_i Th_i - \sum_{i=1}^{n} \lambda_i Tg \right\|$$

$$\leqslant \sum_{i=1}^{n} \lambda_i \| Th_i - Tg \|.$$

Using the condition on the map and the fact $\sum_{i=1}^{n} \lambda_i = 1$, it is easy to see that the above inequality simplifies to

$$\mu \| g - Tg \| \leqslant \mu r$$

where

$$\mu = \sum_{i=1}^{n} \frac{\lambda_i \| h_i - Tg \|}{\| g - Th_i \| + \| h_i - Tg \|}.$$

Since $\mu = 0$ would imply $h_i = Tg$ for every i, μ must necessarily be different from zero. Thus $\| g - Tg \| \leqslant r$. This together with the fact that $g \in K_1$ shows that $co(TH) \subset H$. In view of the fact that H is closed, G is contained in H. Thus $TG(\subset TH) \subset G$.

This shows that the minimal set K_1 contains exactly one point say $\bar{x}$. The invariance of K_1 under T implies $T^2\bar{x} = \bar{x}$ contrary to the assumption that $T^2x \neq x$ for all x in K. Hence, by Remark 3, there exists a fixed point of T. The uniqueness of the fixed point follows from the condition (*) on the map.

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Reference

Ray, B. K., and Singh, S. P. (1978). Fixed point theorems in Banach spaces. *Indian J. pure appl. Math.*, **9**, No. 3, 216-21.