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COSMOLOGICAL SOLUTIONS FOR THE METRIC REPRESENTING NULL RADIATION

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Exact cosmological solutions corresponding to the metric representing null radiation fields have been obtained for perfect fluid distribution. One of the solutions represents the static Einstein universe.

1. INTRODUCTION

Wyman and Trollope (1965) have shown that the nature of a large class of exact solutions to the Einstein-Maxwell null fields depends on the form of the propagation vector. If the propagation vector of an electromagnetic null field is proportional to the gradient then there exists a coordinate system in which the metric tensor has the form

$$g_{ij} = \begin{pmatrix} 2l & 1 & m & n \\ 1 & 0 & 0 & 0 \\ m & 0 & e^{-2v} & 0 \\ n & 0 & 0 & e^{-2v} \end{pmatrix} \qquad \ldots(1.1)$$

where l, m, n and y are functions of coordinates and the signature of the metric is $(-, +, +, +)$. It may be noted that when $l < 0$, the x^1 takes the role of time-like coordinate.

Later Trollope and Smith (1969) have solved the field equations

$$R_{ij} - \tfrac{1}{2} g_{ij} R + \Lambda g_{ij} = T_{ij} \qquad \ldots(1.2)$$

for the metric (1.1) when the universe is filled with incoherent dust. The solutions obtained by them contain Einstein and Schwarzschild-de Sitter solutions as special cases. In this paper we extend the work to the case when the universe is filled with perfect fluid with world velocity, of particles at rest, being given by

$$u^i = \left(\frac{1}{\sqrt{-2l}}, 0, 0, 0\right). \qquad \ldots(1.3)$$

Two particular solutions, one of which includes the well-known Einstein static universe as a special case, have been obtained under certain assumptions. The other, which

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requires further investigation, has not yet been identified with any model known so far. The study of this new model will be reported later.

2. Field Equations

For the sake of convenience we assume $m = n = 0$, in (1.1), that is, we take

$$g_{ij} = \begin{pmatrix} 2l & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & e^{-2v} & 0 \\ 0 & 0 & 0 & e^{-2v} \end{pmatrix}. \qquad \ldots(2.1)$$

The non-zero Christoffel symbols for (1.1) are given in the appendix.

Equations (1.3) and (2.1) imply

$$u_i = \left(-\sqrt{-2l},\ \frac{1}{\sqrt{-2l}},\ 0,\ 0 \right). \qquad \ldots(2.2)$$

The energy momentum tensor for the perfect fluid is given by

$$T_{ij} = -(p + \rho)\, u_i u_j - p g_{ij} \qquad \ldots(2.3)$$

where ρ and p are the density and the pressure of the fluid and the quantities u_i, as usual, are the covariant components of the velocity of the fluid with respect to the actual coordinate system that is being used.

Hence, (2.3) will reduce to

$$T_{ij} = \begin{pmatrix} 2l\rho & \rho & 0 & 0 \\ \rho & \frac{1}{2l}(\rho + p) & 0 & 0 \\ 0 & 0 & -pe^{-2v} & 0 \\ 0 & 0 & 0 & -pe^{-2v} \end{pmatrix}. \qquad \ldots(2.4)$$

The identity $R = 4\Lambda + 3p - \rho$, reduces (1.2) to the form

$$R_{ij} = T_{ij} + \tfrac{1}{2}(2\Lambda + 3p - \rho). \qquad \ldots(2.5)$$

The usual computation leads to the following eight surviving differential equations:

$$-2y_{11} - 2ll_{22} + e^{2v}\nabla^2 l + 2y_1(y_1 - l_2) + 2y_2(l_1 + 2ll_2) = (2\Lambda + 3p + \rho)\, l \qquad \ldots(2.6)$$

$$-2y_{12} + 4y_1y_2 + 2l\{y_{22} - 2(y_2)^2\} + 2y_2l_2 - e^{2v}\nabla^2 y = \tfrac{1}{2}(2\Lambda + 3p - \rho) \qquad \ldots(2.7)$$

$$-2y_{12} - l_{22} + 2y_2(l_2 + y_1) = \tfrac{1}{2}(2\Lambda + 3p + \rho) \qquad \ldots(2.8)$$

$$-2y_{22} + 2(y_2)^2 = \frac{1}{2l}(p + \rho) \qquad \text{...(2.9)}$$

$$(y_1 + l_2)_3 = (y_1 + l_2)_4 = 0 \qquad \text{...(2.10)}$$

$$y_{23} = y_{24} = 0 \qquad \text{...(2.11)}$$

where $\nabla^2 l = l_{33} + l_{44}$ and a suffix after an unknown function denotes partial derivative with respect to the corresponding coordinate.

We further use the transformation

$$u = e^{-\nu} \text{ and } z = u . l. \qquad \text{...(2.12)}$$

Using (2.12), wherever convenient, we consider the following equivalent set of equations:

$$u_2 z_2 = \tfrac{1}{2}(u^2)_{12} - \tfrac{1}{2}\{\nabla^2 y + u^2(\Lambda + p)\} \qquad \text{...(2.13)}$$

$$u_{22} z + u z_{22} = 2uu_{12} - (\Lambda + p)\, u^2 \qquad \text{...(2.14)}$$

$$uu_{11} - 2zu_{12} + \tfrac{1}{2}\nabla^2 l = u_2 z_1 - u_1 z_2 \qquad \text{...(2.15)}$$

$$p + \rho = \frac{4zu_{22}}{u^2} \qquad \text{...(2.16)}$$

$$y_{23} = y_{24} = (y_1 + l_2)_3 = (y_1 + l_2)_4 = 0. \qquad \text{...(2.17)}$$

The compatibility condition $z_{12} = z_{21}$, gives $u_{22} \neq 0$. The resulting equation will be a third order partial differential equation involving only u. From (2.13) and (2.14), we have

$$z = -\tfrac{1}{2}\frac{u}{(u^2)_2}[\nabla^2 y + u^2(\Lambda + p)] + \tfrac{1}{2}\frac{p_2 u^3}{u_2 u_{22}} + \frac{u^2}{u_2}\left(\frac{u_{12}}{u_2} - \frac{u_{122}}{u_{22}}\right). \qquad \text{...(2.18)}$$

We simplify it further by assuming

$$\frac{u_{12}}{u_2} - \frac{u_{122}}{u_{22}} = 0. \qquad \text{...(2.19)}$$

3. Static Non-Empty Universe

We consider the following two cases:

Case I — Let $u_1 = 0$. In view of this eqn. (2.19) is satisfied identically.

From (2.17), we see that

$$y_{23} = y_{24} = 0. \qquad \text{...(3.1)}$$

Hence, we may put

$$u = e^{-\nu} = v(x^2)\, e^{-w} \qquad \ldots(3.2)$$

where $w = w(x^3, x^4)$. When p is constant, eqn. (2.18) becomes

$$z = \tfrac{1}{2} \frac{e^{-w} v}{(v_2)^2} \{e^{2w} \nabla^2 w + v^2(\Lambda + p)\}. \qquad \ldots(3.3)$$

This, in view of (2.13) and (2.14), gives

$$p = e^{2w} \nabla^2 w \frac{v v_{22}}{(v v_2)^2 - v^3 v_{22}} - \Lambda \qquad \ldots(3.4)$$

and

$$p = e^{2w} \nabla^2 w \frac{3(v_2)^2 - v_2 v_{222} - v_{22}(v_2)^2/v}{4v(v_2)^2 v_{22} + v^2 v_2 v_{222} - 3v^2(v_{22})^2 - 2(v_2)^4} - \Lambda. \qquad \ldots(3.5)$$

Since, $e^{2w} \nabla^2 w$ is independent of x^2, one must have

$$\nabla^2 w = K e^{-2w} \qquad \ldots(3.6)$$

where K is non-zero constant. Equation (3.6) is a form of Liouville's equation (Forsyth 1948) and its general solution is known. If now, one compares eqns. (3.4) and (3.5), the result will be

$$v_{22} - cv = 0 \qquad \ldots(3.7)$$

where c is a constant. The general solution of (3.7) is

$$v = A \sin(\sqrt{-c}\, x^2) + B \cos(\sqrt{-c}\, x^2). \qquad \ldots(3.8)$$

Substituting for v in (3.5), we get

$$p = -\frac{K}{A^2 + B^2} - \Lambda. \qquad \ldots(3.9)$$

Again, eqn. (2.16) gives

$$\rho = \frac{3K}{A^2 + B^2} + \Lambda. \qquad \ldots(3.10)$$

Equations (2.12), (3.3) and (3.9) give

$$l = \frac{K}{2c(A^2 + B^2)}. \qquad \ldots(3.11)$$

Hence the solution for this case is given as follows :

$$p = -\frac{K}{A^2 + B^2} - \Lambda,\ \rho = \frac{3K}{A^2 + B^2} + \Lambda$$

$$l = \frac{K}{2c(A^2 + B^2)},\ e^{-\nu} = v(x^2)\, e^{-w}$$

(continued on p. 547)

$$\left.\begin{aligned} K &= e^{2w} \nabla^2 w \\ v &= A \sin(\sqrt{-c}\, x^2) + B \cos(\sqrt{-c}\, x^2) \\ e^{-2w} &= -\left[\frac{4F'(Z)\, G'(\bar{Z})}{K\{F(Z) + G(\bar{Z})\}^2}\right] \end{aligned}\right\} \quad \text{...(3.12)}$$

where A and B are arbitrary real constants and F and G are arbitrary but non-constant functions of $Z\,(= x^3 + ix^4)$ and $\bar{Z}\,(= x^3 - ix^4)$ respectively. By the method similar to the one given by Trollope and Smith (1969) the line element corresponding to the solutions (3.12) can be reduced to the form

$$ds^2 = -\,dt^2 + \frac{dr^2}{1 - (r^2/R^2)} + r^2(d\theta^2 + \sin^2\theta\, d\Phi^2) \quad \text{...(3.13)}$$

which is the usual form of Einstein static universe.

Case II : *More non-static empty universes* — We now assume $u_2 = 0$. The field eqns. (2.13) - (2.17) reduce to

$$\left.\begin{aligned} \nabla^2 y &= -(\Lambda + p)\, e^{-2v} \\ l_{22} &= -(\Lambda + p) \\ u(u_{11} + u_1 l_2) + \tfrac{1}{2}\nabla^2 l &= 0 \\ (y_1 + l_2)_3 = (y_1 + l_2)_4 &= 0. \end{aligned}\right\} \quad \text{...(3.14)}$$

After some straightforward manipulation we are led to the following general solution (p is constant):

(i) when $\Lambda \neq 0$ and $p \neq 0$ we get

$$\left.\begin{aligned} e^{-2v} &= \frac{4F_1' G_1'}{(\Lambda + p)(F_1 + G_1)^2} \\ l &= -\tfrac{1}{2}(\Lambda + p)(x^2)^2 + (h - y_1)\, x^2 + \frac{1}{\Lambda + p} \\ &\quad \times (y_{11} + h y_1) + f + g. \end{aligned}\right\} \quad \text{...(3.15)}$$

(ii) when $\Lambda = 0$ and $p = 0$, we get

$$\left.\begin{aligned} y &= F_1 + G_1 \\ l &= (h - y_1)\, x^2 - \iint [(e^{-2v})_{11} + h(e^{-2v})_1]\, dZ\, d\bar{Z} \end{aligned}\right\} \quad \text{...(3.16)}$$

where $F_1 = F_1(Z, x^1)$, $G_1 = G_1(\bar{Z}, x^1)$, $f = f(Z)$, $g = g(\bar{Z})$ and $h = h(x^1)$ are arbitrary functions.

The primes on F_1 and G_1 denote differentiation with respect to Z and $\bar{Z}$ respectively. From the viewpoint of physics, which will be reported later, the solutions obtained here are more general than those obtained by Trollope and Smith (1969).

References

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Appendix

$$\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = -\, l_2, \quad \begin{Bmatrix} 3 \\ 11 \end{Bmatrix} = -\, e^{2\nu} l_3, \quad \begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 44 \end{Bmatrix} = e^{-2\nu} y_2,$$

$$\begin{Bmatrix} 4 \\ 11 \end{Bmatrix} = -\, e^{2\nu} l_4, \begin{Bmatrix} 2 \\ 11 \end{Bmatrix} = l_1 + 2 l l_2, \begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 14 \end{Bmatrix} = -\, y_1,$$

$$\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = l_2, \quad \begin{Bmatrix} 2 \\ 14 \end{Bmatrix} = l_4, \quad \begin{Bmatrix} 3 \\ 23 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 24 \end{Bmatrix} = -\, y_2,$$

$$\begin{Bmatrix} 2 \\ 13 \end{Bmatrix} = l_3, \quad \begin{Bmatrix} 3 \\ 33 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 34 \end{Bmatrix} = -\begin{Bmatrix} 3 \\ 44 \end{Bmatrix} = -\, y_3,$$

$$\begin{Bmatrix} 2 \\ 33 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 44 \end{Bmatrix} = e^{-2\nu}(y_1 - 2 l y_2), \quad \begin{Bmatrix} 3 \\ 34 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 44 \end{Bmatrix} = -\begin{Bmatrix} 4 \\ 33 \end{Bmatrix} = -\, y_4.$$