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ON *GF*-HYPERSURFACES

H. B. PANDEY

Mathematics Department, R.B.S. College, Agra

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Yano and Ishihara (1968) studied complex hypersurfaces in a Kählerian manifold with constant holomorphic sectional curvature. In the present note the author has taken *GF*-hypersurface in a *K*-manifold. *GF*-hypersurface is more general than complex hypersurface. Several properties of *GF*-hypersurface have been studied.

INTRODUCTION

Let M be a $(n + 2)$-dimensional differentiable manifold. Let in M there be a tensor field F of type (1, 1) satisfying

$$F^2 = a^2 I \quad \text{or} \quad F^A_B F^B_C = a^2 \delta^{A^*}_C \qquad \text{...(1.1)}$$

where a** is a constant real or imaginary and I is identity. Then F is called a *GF*-structure and M is said to be a *GF*-structure manifold (Duggal 1971).

Let on M, a Riemannian metric G be introduced satisfying

$$F^E_C F^D_B G_{ED} = -a^2 G_{CB}. \qquad \text{...(1.2)}$$

Then the structure (F, G) is called H-structure on M and M is H-structure manifold. If in addition to above two conditions, the following

$$\nabla_C F^A_B = 0 \qquad \text{...(1.3)}$$

also holds good, then M is called K-manifold. If we put

$$F_{CB} = F^D_C G_{DB} \qquad \text{...(1.4a)}$$

we easily see that

$$F_{CB} + F_{BC} = 0. \qquad \text{...(1.4b)}$$

Let there be given in M a differentiable submanifold V of class C^∞ and of dimension n. Let V be expressed in each neighbourhood of M by equations†

*A, B, C, D ... will run from 1 to $n + 2$.

**In the sequel we take $a^4 = 1$.

†a, b, c, d ... will run from 1 to n.

$$X^A = X^A(u^a) \qquad ...(1.5)$$

where (X^A) are local coordinates of M in $\bar{U}$ and (u^a) local coordinates of V in $U = \bar{U} \cap V$. We have in U local vector fields B_b having components

$$B_b^A = \partial_b X^A \qquad ...(1.6)$$

and spanning the tangent space of V at each print of U, where ∂_b denotes the operator $\dfrac{\partial}{\partial u^b}$.

The submanifold V is a GF-hypersurface when and only when the GF-structure F leaves invariant the tangent spaces of V at each point of V. In the sequel, we shall restrict ourselves only to GF-hypersurfaces. F or a GF-hypersurface V, FB_b is a linear combination of B_a in U that is

$$FB_b = f_b^a B_a \text{ is } F_B^A B_b^B = f_b^a B_a^A \qquad ...(1.7)$$

where the functions f_b^a are components of a tensor field f of type (1, 1) defined globally in V. Applying the operator F to both the sides of (1.7) and taking account of (1.1) we get

$$f^2 = a^2 I, \quad \text{i.e.} \quad f_b^a f_c^b = a^2 \delta_c^a . \qquad ...(1.8)$$

The Riemannian metric g induced in V has components of the form

$$g_{cb} = G_{CB} B_c^C B_b^B \qquad ...(1.9)$$

in each neighbourhood U of V. Thus we get

$$f_c^e f_b^d g_{ed} = -a^2 g_{cb}. \qquad ...(1.10)$$

Let us put

$$f_{cb} = f_c^e g_{eb}. \qquad ...(1.11)$$

Then, we have

$$f_{cb} + f_{bc} = 0 \qquad ...(1.12)$$

by virtue of (1.4*b*).

Since V is a GF-hypersurface, the normal plane of V is left invariant by the GF-structure F of M at each point of V. Thus there exist in each neighbourhood U of V two local unit vector fields C and D normal to V, such that

$$FC = iaD;\ FD = -iaC \qquad \ldots(1.13)$$

where C and D being perpendicular to each other.

We have, as is well known, the following equations:

$$\left.\begin{aligned} \nabla_c B_b^A &= h_{cb} C^A + k_{cb} D^A \\ \nabla_c C^A &= -\, h_c^a\, B_a^A + l_c D^A \\ \nabla_c D^A &= -\, k_c^a\, B_a^A - l_c C^A. \end{aligned}\right\} \qquad \ldots(1.14)$$

These equations are Gauses and Weingarten formulae for the *GF*-hypersurface V. The left-hand sides of these equations are defined by

$$\nabla_c B_b^A = \partial_c B_b^A + \left\{ {A \atop CB} \right\} B_c^C B_b^B - \left\{ {a \atop cb} \right\} B_a^A$$

$$\nabla_c C^A = \partial_c C^A + \left\{ {A \atop CB} \right\} B_c^C C^B$$

$$\nabla_c D^A = \partial_c D^A + \left\{ {A \atop CB} \right\} B_c^C D^B$$

respectively, where $\left\{ {A \atop CB} \right\}$ and $\left\{ {a \atop cb} \right\}$ are Christoffel symbols determined respectively by G_{CB} and g_{cb}. The functions h_b^a and k_b^a appearing in (1.14) are the components of second fundamental tensors h and k respectively, where h and k are local tensor fields of type (1, 1) defined in each neighbourhood U of V with respect to the choice of the unit normal vector fields C and D. The functions and h_{cb} and k_{cb} are defined as:

$$h_{cb} = h_c^a\, g_{ab},\ k_{cb} = k_c^a\, g_{ab}. \qquad \ldots(1.15)$$

We can see that

$$\begin{aligned} h_{cb} = h_c^a\, g_{ab} &= h_b^a\, g^{bd} g_{dc} g_{ab} \\ &= h^{ad} g_{dc} g_{ab} \\ &= h_b^d\, g_{dc} \\ &= h_{bc}. \end{aligned} \qquad \ldots(1.16)$$

Similarly

$$k_{cb} = k_{cb}. \qquad \ldots(1.17)$$

Here l is third fundamental tensor which is a local covector field.

Differentiating (1.7) covariantly along V and taking account of (1.13) and (1.14) we get

$$\nabla_c f_b^a = 0 \quad \text{...(1.18)}$$

and

$$iah_{cb} = k_{ce} f_b^e \; iak_{cb} = -h_{ca} f_b^a \quad \text{...(1.19)}$$

which is equivalent to

$$iah = kf \quad \text{or} \quad iah_c^a = k_c^e f_a^e \quad \text{...(1.20)}$$

$$iak = -hf \quad \text{or} \quad iak_c^a = -h_c^e f_e^a \quad \text{...(1.21)}$$

which imply that

$$h_e^e = 0, \; k_e^e = 0 \quad \text{...(1.22)}$$

$$fh + hf = 0 \quad \text{and} \quad fk + kf = 0. \quad \text{...(1.23)}$$

Moreover

$$h^2 = k^2 \quad \text{...(1.24)}$$

$$hk + kh = 0 \quad \text{...(1.25)}$$

and

$$f_c^e f_b^d \gamma_{ed} = \gamma_{cb}, \; \gamma_{cb} = \gamma_{bc} \quad \text{...(1.26)}$$

where we have put

$$\gamma_{cb} = h_c^e h_{be}. \quad \text{...(1.27)}$$

If we take account of (1.8), (1.10), (1.18) and (1.22) we see that any GF-hypersurface in a K-manifold is a minimal surface and is itself a K-manifold with the induced K-structure (f, g).

Let us take two intersecting neighbourhoods U and $\bar{U}$ of V. We can choose pairs (C, D) and $(\bar{C}, \bar{D})$ of normal vector fields defined respectively in U and $\bar{U}$ such that they are related to each other by

$$\bar{C} = c \cos \theta - D \sin \theta; \; \bar{D} = -C \sin \theta + D \cos \theta, \text{ in } U \cap \bar{U} \quad \text{...(1.28)}$$

θ being certain function in $U \cap \bar{U}$.

In the light of (1.13) one can check that $F\bar{C} = ia\bar{D}$ and $F\bar{D} = -ia\bar{C}$. If we denote by $\bar{h}$, $\bar{k}$ and $\bar{l}$ respectively the second and the third fundamental tensors in U w.r.t. C and D, then we easily obtain in $U \cap \bar{U}$

$$\bar{h} = h \cos\theta - k \sin\theta,\ \bar{k} = h \sin\theta + k \cos\theta;\ \bar{l} = l - d\theta. \qquad ...(1.29)$$

Thus we get

$$h^2 = \bar{h}^2 = k^{-2} = k^2;\ \bar{h}\bar{k} = hk = kh = \bar{k}\bar{h},\ d\bar{l} = dl \qquad ...(1.30)$$

in $U \cap \bar{U}$. Consequently we have :

Proposition 1.1 — For any GF-hypersurface V in a K-manifold h^2, k^2, hk, kh and $\Omega = dl$ determine global tensor field of corresponding type in V, respectively. They satisfy

$$h^2 = k^2;\ hk = -kh.$$

The local tensor field γ_{cb} defined in (1.26) determines a global tensor field of type (0, 2) in V.

Let us write the structure equations for a GF-hypersurface V in a K-manifold:

$$'K_{DCBA}B^D_d\, B^C_c\, B^B_b\, B^A_a = K_{dcba} - (h_{da}h_{cb} - h_{ca}h_{db} - (k_{da}k_{cb} - k_{ca}k_{db}) \qquad ...(1.31)$$

$$'K_{DCBA}B^D_d\, B^C_c\, B^B_b\, C^A = \nabla_d h_{cb} - \nabla_c h_{db} - l_d k_{cb} + l_c k_{db} \qquad ...(1.32)$$

$$'K_{DCBA}B^D_d\, B^C_c\, B^B_b\, D^A = \nabla_d k_{cb} - \nabla_c k_{db} + l_d h_{cb} - l_e h_{db} \qquad ...(1.33)$$

$$'K_{DCBA}B^D_d\, B^C_c\, C^B D^A = \nabla_d l_c - \nabla_c l_d + H^e_d\, k_{ec} - h^e_c\, k_{ei} \qquad ...(1.34)$$

where $'K_{DCBA}$ and k_{dcba} are components of the curvature tensor of the enveloping manifold M and the GF-hypersurface V respectively.

2. GF-Hypersurface in a K-Manifold With Certain Constant

Let in M the curvature tensor satisfy

$$'K_{DCBA} = \frac{C}{4}\,[(G_{DA}G_{CB} - G_{CA}G_{DB}) + (F_{DA}F_{CB} - F_{CA}F_{DB} - 2F_{DC}F_{BA})] \qquad ...(2.1)$$

Then the K-manifold M will be called a manifold M with constant C. Substituting (2.1) in (1.31)-(1.34) one gets

$$K_{dcba} = \frac{C}{4}\,[(g_{da}g_{cb} - g_{ca}g_{db}) + (f_{da}f_{cb} - f_{ca}f_{db} - 2f_{dc}f_{ba})]$$
$$+ (h_{da}h_{cb} - h_{ca}h_{db}) + (k_{da}k_{cb} - k_{ca}k_{db}) \qquad ...(2.2)$$

$$\nabla_d h_{cb} - \nabla_c h_{db} - l_d k_{cb} + l_c k_{db} = 0 \quad ...(2.3)$$

$$\nabla_d k_{cb} - \nabla_c k_{db} + l_d h_{cb} - l_c h_{db} = 0 \quad ...(2.4)$$

$$\nabla_d l_c - \nabla_c l_d + h_d^e k_{ce} - h_c^e k_{de} + \frac{Cia}{2} f_{dc} = 0. \quad ...(2.5)$$

Transvecting (2.1) with g^{da} and taking account of (1.27) we get

$$K_{cb} = \frac{C}{4}(n - 1 - 3a^2)\, g_{cb} - 2\gamma_{cb}. \quad ...(2.6)$$

Since $a^4 = 1$, the maximum root of a is $+1$. Hence, let $n > 4$ then $n - 1 - 3a^2 > 0$. Thus for a *GF*-hypersurface V in a *K*-manifold M with non positive constant C, the Ricci form of V satisfies the inequality $K_{cb}X^c X^b \leqslant 0$ for any values of variables X^a. Let $n = 4$ and $a^2 = -1$. Then *GF*-hypersurface beomes complex hypersurface and *K*-manifold with constant C becomes Kählerian manifold with constant holomorphic sectional curvature C for which the above inequality always holds (Yano and Ishihara 1968). Let $n = 4$ and $a = 0$. In this case one can easily verify the above inequality. Thus for $n \geqq 4$ one can check that $k_{cb} = 0$ if and only if $C = 0$ and $\gamma_{cb} = 0$. Thus we have the following.

Proposition 2.1 — For a *GF*-hypersurface V in a *K*-manifold M with dimension $n \geqslant 4$ with non positive constant C, the Ricci form of V satisfies the inequality $K_{cb} X^c X^b \leqslant 0$ for any values of variables X^a. In this case, the equality $K_{cb} = 0$ holds identically if and only if $C = 0$ and V is totally geodesic.

Similarly we can analyse the case when $n < 4$. Let the *GF*-hypersurface in a *K*-manifold with constant C be Einstein manifold. Then

$$\gamma_{cb} = \tfrac{1}{2}\left[\frac{cn(n - 1 - 3a^2) - 4K}{4n}\right] g_{cb}. \quad ...(2.7)$$

Here $K < cn(n - 1 - 3a^2)/4$. Thus we have the following.

Proposition 2.2 — Let V, a *GF*-hypersurface in a *K*-manifold M with constant C be Einstein manifold. Then the scalar curvature K of V satisfies the inequality $K < cn(n - 1 - 3a^2)/4$ where dim $V = n$. In this case equality holds if and only if V is totally geodesic.

Let $\nabla_d K_{cb} = 0$. Then from (2.6) we have $\nabla_b(h_c^e h_{be}) = 0$. When V is irreducible as a Riemannian manifold, we have

$$h_c^e h_{be} = A g_{cb} \quad ...(2.8)$$

A being a constant. Thus V is an Einstein manifold if V is irreducible and $V_d K_{cb} = 0$. When V is reducible and not locally flat taking an arbitrary coordinate neighbourhood U of V we see that there exists an irreducible factor U_1 of U in the so-called de-Rham decomposition of U. Thus U is pythagorean product $U_1 \times U_2$ where U_1 and U_2 are two local K-manifolds. Let $(u^1, u^2, u^3, \ldots, u^n)$ and $u^{r+1}, u^{r+2}, \ldots, u^{n+2})$ be coordinates defined in U_1 and in U_2 respectively. Then we have at any point of U*

$$g_{\alpha\lambda} = 0, \; f_{\alpha\lambda} = 0 \qquad \ldots(2.9)$$

and

$$h_{\alpha}^{e} \, h_{\lambda e} = 0. \qquad \ldots(2.10)$$

Thus $$h_{\alpha\lambda} = 0, \; k_{\alpha\lambda} = 0. \qquad \ldots(2.11)$$

Put $a = \alpha,\ b = \beta,\ c = \lambda,\ d = \mu$ in (2.2) we get

$$K_{\mu\lambda\beta\alpha} = -\, 2 f_{\mu\lambda} f_{\beta\alpha} \qquad \ldots(2.12)$$

But $K_{\mu\lambda\beta\alpha} = 0$ because U is a pythagorean product $U_1 \times U_2$. Thus contradicts (2.12). Hence V is necessarily irreducible. When V is locally flat it is obviously Einstein manifold. Thus

Proposition 2.3 — A GF-hypersurface of a K-manifold with constant C given is an Einstein manifold if and only if $\nabla_d K_{cb} = 0$ is satisfied.

Now operating (2.3) and (2.4) by f_{e}^{c} we get:

$$f_{e}^{c} \, (\nabla_d h_{cb} - \nabla_c h_{db}) - f_{e}^{c} \, (l_d k_{cb} - l_c k_{db}) = 0 \qquad \ldots(2.13)$$

and

$$f_{e}^{c} \, (\nabla_d k_{cb} + l_d h_{cb}) - f_{e}^{c} \, (\nabla_c k_{db} + l_c h_{db}) = 0. \qquad \ldots(2.14)$$

Using in (2.13) and (2.14) eqns. (1.18), (1.20) and (1.21) we get

$$ia(\nabla_d k_{eb} + l_d h_{eb}) + f_{e}^{e} \, (\nabla_c h_{dn} l_c k_{db}) = 0 \qquad \ldots(2.15)$$

$$ia(\nabla_d h_{en} - l_d k_{eb}) - f_{e}^{e} \, (\nabla_c k_{db} + l_c h_{db}) = 0. \qquad \ldots(2.16)$$

Thus eqns. (2.15) and (2.16) imply

$$(\nabla_d h_{cb} - l_d k_{eb}) - a^2 f_{d}^{e'} \, f_{e}^{c} \, (\nabla_{e'} h_{cb} - l_{e'} k_{cb}) = 0 \qquad \ldots(2.17)$$

which gives

$$\nabla_d h_{eb} = l_d k_{eb} + a^2 f_{d}^{e'} \, f_{e}^{c} \, (\nabla_e h_{cb} - l_e k_{cb}). \qquad \ldots(2.18)$$

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*$1 \leqslant \alpha, \beta, \gamma, \ldots \leqslant r$ and $r + 1 \leqslant \lambda, \mu, \nu \leqslant n + 2$.

Now let $\nabla_d k_{cb} = 0$. Then it is equivalent to $\nabla_d \gamma_{cb} = 0$ or $\nabla_d h^e_c h_{eb} = 0$ or to

$$(\nabla_d h_{cb}) h^b_a + h^b_c (\nabla_d h_{ba}) = 0. \qquad ...(2.19)$$

Putting the value of $\nabla_d h_{cb}$ from (2.18) in (2.19) we get

$$[l_d k_{eb} + a^2 f^{e'}_d f^c_e (\nabla_{e'} h_{cb} - l_{e'} k_{cb})] h^b_a$$
$$+ h^b_e [l_d k_{ba} + a^2 f^{e'}_d f^c_b (\nabla_{e'} h_{ca} - l_{e'} k_{ca})] = 0. \qquad ...(2.20)$$

Now in the light of (1.25) we get

$$f^c_e (\nabla_{e'} h_{cb} - l_{e'} k_{cb}) h^b_a + f^c_b h^b_e (\nabla_{e'} h_{ca} - l_{e'} k_{ea}) = 0 \qquad ...(2.21)$$

which reduces to

$$(\nabla_{e'} h_{cb} - l_{e'} k_{cb}) h^b_a = 0. \qquad ...(2.22)$$

Conversely, let (2.22) hold, then

$$(\nabla_{e'} h_{cb}) h^b_a + h^b_a (l_{e'} k_{cb}) = 0. \qquad ...(2.23)$$

From (2.18), (2.22) and (2.23) we get

$$(\nabla_{e'} h_{cb}) h^b_a + h^b_e (\nabla_{e'} h_{ba}) = 0 \qquad ...(2.24)$$

which is equivalent to $\nabla_d K_{cb} = 0$. Hence we have

Lemma 2.1 — For a *GF*-hypersurface in a *K*-manifold *M* with constant *C* the following conditions are equivalent to each other:

(*a*) $\nabla_d K_{cb} = 0$, (*b*) $(\nabla_d h_{cb} - l_d k_{cb}) h^b_a = 0$,

(*c*) $(\nabla_d k_{cb} + l_d h_{cb}) k^b_a = 0$.

Let the *GF*-hypersurface *V* be Einstein manifold.

Then $\nabla_d K_{cb} = 0 \Rightarrow \nabla_d h_{cb} = l_d k_{cb}$ and $\nabla_d k_{cb} = - l_d h_{cb}$.

Differentiating (2.1) covariantly and taking these into account one can have

$$\nabla_e K_{dcba} = 0.$$

Thus we have the following.

Theorem 2.1 — For a *GF*-hypersurface *V* of a *K*-manifold *M* with constant *C* the following conditions are equivalent:

(*a*) V is an Einstein manifold,

(*b*) the Ricci tensor of V is parallel,

(*c*) V is locally symmetric.

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