Indian J. pure appl. Math., 10(5) : 558-566, May 1979

FINITE TIME STABILITY OF STOCHASTIC CONTROL SYSTEMS

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(*Received* 3 *September* 1978)

The present paper is concerned with the various types of finite time stabilities in the mean of a stochastic control system of the form

$$x'(t; \omega) = f(t, x(t; \omega); \omega) + \phi(t, u(t; \omega); \omega)$$

under some suitable conditions on the functions f and ϕ. The main results involve the existence of Lyapunov-like functions which in general do not possess the usual definiteness requirements on such functions or their derivative.

1. INTRODUCTION

Recently stochastic integrodifferential equations with operator coefficients have been studied by a number of authors (see Reid 1972, Ladde *et al.* 1974, Rao and Manongian 1977, Tsokos and Padgett 1971, Tsokos and Lakshmikantham 1968). Many recent papers have been dealt with the stability, boundedness and other properties of stochastic differential and integral equations in the mean (see Tsokos and Rao 1969, Weiss and Infante 1967). Here we wish to study the finite time stability of the stochastic control system of the form

$$x'(t; \omega) = f(t, x(t; \omega); \omega) + \phi(t, u(t; \omega); \omega) \qquad \text{...(1)}$$

where $\quad f \in C[I \times R^n \times \Omega, R^n], \phi \in C[I \times R^m \times \Omega, R^m]$

in which I denotes the interval $[t_0, t_0 + T)$ and let $\bar{I}$ denote the interval $[t_0, t_0 + T]$, R^n be the real Euclidean n-space and $\omega \in \Omega$, the supporting set of a complete probability measure space (Ω, F, P).

Let $y(t; \omega)$ be any continuous differentiable function for $t \in I$, $\omega \in \Omega$ which takes values from R^n. The problem of finite time stability for ordinary differential equations and control systems have been studied by Weiss and Infante (1967) and Tsokos and Rao (1969). In this paper we extend the definitions of various types of stability in the mean over a finite time interval in stochastic sense and employ the concept of Lyapunov's like functions and the theory of integral inequalities. In section 3 we study the stability, uniform stability, quasi-expansive stability and quasi-contractive stability in the mean of a stochastic control system (1).

2. Preliminaries

Let $y(t;\omega)$ be the given target function. Consider the transformation

$$\begin{aligned} z(t;\omega) &= x(t;\omega) - y(t;\omega) \\ z'(t;\omega) &= x'(t;\omega) - y'(t;\omega) \\ &= f(t, x(t;\omega);\omega) + \phi(t, u(t;\omega);\omega) - y'(t;\omega) \end{aligned}$$

where $u(t;\omega)$ is a control function defined for $t \in I$.

Define
$$\left.\begin{aligned} F(t, z(t;\omega);\omega) &= f(t, z(t;\omega) + y(t;\omega);\omega) - f(t, y(t;\omega);\omega) \\ \psi(t, y(t;\omega);\omega) &= y'(t;\omega) - f(t, y(t;\omega);\omega) \\ \xi(t;\omega) &= \phi(t, u(t;\omega);\omega) - \psi(t, y(t;\omega);\omega). \end{aligned}\right\} \quad ...(2)$$

Then the stochastic control system reduces to

$$z'(t;\omega) = F(t, z(t;\omega);\omega) + \xi(t;\omega). \quad ...(3)$$

Noting that $F(t, 0;\omega) = 0$. Above considerations will imply that it is enough to study the stability properties of the reduced stochastic control system (3) with respect to the set $z(t;\omega) = 0$. We, therefore, will concentrate on stochastic control system (3).

The function $V = V(t, z(t;\omega);\omega) : \bar{I} \times R^n \times \Omega \to R^n$ is said to belong to the class $\triangle$ if it is continuous in both t and $z(t;\omega)$ and satisfies for each $t \in \bar{I}$ a local Lipschitz condition in $z(t;\omega)$. The function $g(t; r(t))$, $\bar{I} \times R' \to R'$ is said to belong to the class Γ if it is continuous in both t and $r(t)$, monotonic increasing in $r(t)$ for each $t \in I$ and it is smooth enough to ensure the existence of a maximal solution of the differential equation.

$$r'(t) = g(t, r + n(t)),\ r(t_0) = r_0 \quad ...(4)$$

where $n(t)$ is a continuous function over $\bar{I}$. We further use the following notations:

$$B(a) = \{z : z \in R^n, \| z \| < a\}$$

$$\bar{B}(a) = \{z : z \in R^n, \| z \| \leqslant a\}$$

$$V_m^\alpha(t) = \operatorname*{Inf}_{\| z \| = \alpha} E[V(t, z(t;\omega);\omega)]$$

$$V_M^\alpha(t) = \sup_{\| z \| < \alpha} E[V(t, z(t;\omega);\omega]$$

$$V^\alpha(t) = \sup_{\| z \| < \alpha} E[V(t, z(t;\omega);\omega]$$

$$V_\delta^\alpha(t) = \sup_{z \in B(\alpha) - B(\delta)} E[V(t, z(t;\omega);\omega)], \quad \text{with } \delta < \alpha$$

and

$$D^+E[V(t, z(t;\omega);\omega)] = \lim_{h\to 0^+} \sup \frac{1}{h} [E\{V(t+h, z(t;\omega) + hF(t, z(t;\omega);\omega)\}/z(t;\omega) - V(t, z(t;\omega);\omega)] \quad \ldots(5)$$

where F is given function in (3). Secondly for every pair of numbers α, β, such that $0 < \alpha < \beta$, we have,

$$\| V(t, z(t;\omega);\omega) - V(t, \bar{z}(t;\omega);\omega) \| \leqslant M \| z(t;\omega) - \bar{z}(t;\omega) \| \quad \ldots(6)$$

$$M > 0, \text{ for } t \in \bar{I} \text{ and } z, \bar{z} \in \bar{B}(\beta) - B(\alpha).$$

For our subsequent discussion we present the following definitions.

Let $z(t;\omega) = z(t, t_0, z_0(\omega))$ be any solution of the stochastic control system (3) through $(t_0, z_0(\omega))$.

Definition 1 — The stochastic control system (3) is said to be stable in the mean with respect to $(\alpha, Y, t_0, T, \|\cdot\|)$, $0 < \alpha \leqslant Y$ if for every trajectory $z(t;\omega)$ of (3) the condition $\| z(t_0;\omega) \| < \alpha$ implies $E[\| z(t;\omega) \|] < Y$ for all $t \geqslant t_0$, $t \in I$.

Definition 2 — The stochastic control system (3) is said to be uniformly stable in the mean with respect to $(\alpha, Y, t_0, T, \|\cdot\|)$ $0 \leqslant \alpha \leqslant Y$, if for every trajectory $z(t;\omega)$ of (3), the condition $\| z(t;\omega) \| < \alpha$ implies $E[\| z(t;\omega)\|] < Y$, for all $t \geqslant t_1$, $t, t_1 \in I$.

Definition 3 — The stochastic control system (3) is said to be quasi-expansively stable in the mean with respect to $(\alpha, \beta, t_0, T, \|\cdot\|)$, $\alpha \leqslant \beta$, if for every trajectory $z(t;\omega)$ of (3) the condition $\| z(t_0;\omega) \| < \alpha$, implies that there exists a $t_1 \in I$, such that $E[\| z(t;\omega) \|] < \beta$ for $t \in (t_1, t_0 + T)$.

Definition 4 — The stochastic control system (3) is said to be expansively stable in the mean with respect to $(\alpha, \beta, Y, t_0, T, \|\cdot\|)$, $\alpha \leqslant \beta < Y$, if it is stable in the mean with respect to $(\alpha, Y, t_0, T, \|\cdot\|)$ and it is quasi-expansively stable in the mean with respect to $(\alpha, \beta, t_0, T, \|\cdot\|)$.

Definition 5 — The stochastic control system (3) is said to be quasi-contractively stable in the mean with respect to $(\alpha, \beta, t_0, T, \|\cdot\|)$, $\alpha > \beta$, if for every trajectory $z(t;\omega)$ of (3) the condition $\| z(t_0;\omega) \| < \alpha$, implies that there exists a $t_1 \in I$ such that $E[\|z(t;\omega)\|] < \beta$ for $t \in (t_1, t_0 + T)$.

Definition 6 — The stochastic control system (3) is said to be contractively stable in the mean with respect to $(\alpha, \beta, Y, t_0, T, \|\cdot\|)$, $\beta < \alpha \leqslant Y$, if it is stable in the mean with respect to $(\alpha, Y, t_0, T, \|\cdot\|)$ and it is quasi-contractively-stable in the mean with respect to $(\alpha, \beta, t_0, T, \|\cdot\|)$.

We require the following Lemmas in our future discussion.

Lemma 1 — If $u(t) \leqslant n(t) + \int_{t_0}^{t} g(s, u(s))\, ds + r(t_0)$ where $g \in \Gamma$, $u(t)$ and $n(t)$ are continuous functions for $t \in I$, then $u(t) \leqslant n(t) + r(t)$, where $r(t)$ is the maximal solution of (4) through $(t_0, r(t_0))$.

Lemma 2 — Assume that there exists a measurable map $V(t, z(t; \omega); \omega)$ from $I \times R^n \times \Omega$ into I such that

(i) $\| V(t, z(t; \omega); \omega) - V(t, \bar{z}(t; \omega); \omega) \| \leqslant \lambda \| z(t; \omega) - \bar{z}(t; \omega) \|$,

for $z, \bar{z} \in R^n$, $t \in I$ and $w \in \Omega$;

(ii) V is continuous in t and $E[V(t, 0; \omega)] = 0$;

(iii) The function $D^+E[V(t, z(t; \omega); \omega)]$ defined in (5) satisfies the condition

$$D^+E[V(t, z(t; \omega); \omega)] \leqslant g(t, E[V(t, z(t; \omega); \omega)]) \qquad \ldots(7)$$

for all $t \in I$, $z \in \bar{B}(\beta) - B(\alpha)$ and $g \in \Gamma$ where $g(t, r)$ is concave in r for each $t \in I$;

(iv) For each $t_0 \in I$, $r(t, t_0, r_0)$ is the maximal solution of the scalar initial value problem

$$r'(t) = g(t, r(t)),\ r(t_0) = r_0$$

which exist for all $t \geqslant t_0$.

Further

$$u(t; \omega) = M \int_{t_0}^{t} \| \xi(s; \omega) \|\, ds, \quad t \in \bar{I} \qquad \ldots(8)$$

where $\xi(t; \omega)$ is the same function as defined in (2) and M a constant greater than zero. Let $z_1 \in B(\beta) - B(\alpha)$ and $z(t, t_1, z_1)$ denote any solution of (3). Then

$$r_1 = V_M^{\|z_1\|}(t_1; \omega) - u(t_1; \omega) \qquad \ldots(9)$$

implies that

$$E[V(t, z(t, t_1, z_1); \omega)] \leqslant r(t, t_1, r_1) + u(t; \omega) \qquad \ldots(10)$$

for $t \geqslant t_1$ as long as $z(t, t_1, z_1) \in \bar{B}(\beta) - B(\alpha)$, where $r(t, t_1, r_1)$ denotes the maximal solution of the scalar differential equation

$$r'(t) = g(t, r(t) + u(t)) \qquad \ldots(11)$$

through (t_1, r_1)

PROOF : Let $z_1 \in B(\beta) - B(\alpha)$ and $z(t, t_1, z_1) \in B(\beta) - B(\alpha)$ then for sufficiently small $h > 0$

$$z(t + h, t_1, z_1) \in \bar{B}(\beta) - \beta(\alpha).$$

Denote $z(t, t_1, z_1)$ by $z(t; \omega)$.

For $m(t) = E[V(t, z(t; \omega); \omega]$ we have

$$\begin{aligned} m(t+h) - m(t) = E[&V(t+h; z(t+h; \omega); \omega) \\ &- V(t+h, z(t; \omega) + hF(t, z(t; \omega); \omega) + h\xi(t; \omega)) \\ &+ V(t+h, z(t; \omega) + hF(t, z(t; \omega); \omega) + h\xi(t; \omega)) \\ &- V(t+h, z(t; \omega) + hF(t, z(t; \omega); \omega)) \\ &+ V(t+h, z(t; \omega) + hF(t, z(t; \omega); \omega)) \\ &- V(t, z(t; \omega); \omega)]. \end{aligned}$$

Then in view of (5), (6) and (7) we obtain

$$\lim_{h \to 0^+} \sup \left[\frac{m(t+h) - m(t)}{h} \right] \leqslant g(t, m(t)) + M.E[\| \xi(t; \omega) \|].$$

Integrating both sides with respect to t between t_1 and t and using (8), we have

$$m(t) \leqslant m(t_1) - u(t_1 : \omega) + u(t; \omega) + \int_{t_1}^{t} g(s, m(s))\, ds.$$

Now by applying Lemma 1 we obtain the required result.

3. MAIN RESULTS

In this section we state and prove the theorems which yield sufficient conditions for stability, uniform stability, quasi-expansive stability and quasi-contractive stability in the mean of the stochastic control system (3). Our first theorem deals with the finite stability in the mean of the stochastic control system (3) under some suitable conditions on the functions V and g.

Theorem 1 — The stochastic control system (3) is finitely stable in the mean with respect to $(\alpha, Y, t_0, T, \|\cdot\|)$, $\alpha < Y$, if there exist functions V and g such that

(i) For $t \in I$, $z \in \bar{B}(Y)$, (7) and (8) hold, and

(ii) The maximal solution $r(t, t_0, r(t_0))$ of (11) with $r(t_0) = -U(t_0; \omega) + V^{\alpha}(t_0)$ satisfies the condition

$$r(t, t_0, r(t_0)) < V_m^{Y}(t) - U(t; \omega) \quad \text{for all } t \geqslant t_0. \qquad \ldots(12)$$

PROOF : Let $z(t; \omega)$ denote any arbitrary trajectory of (3) with $\| z(t_0; \omega) \| \leqslant \alpha$. Let there exist a $t_1 \in I$ such that $E[\| z(t_1; \omega) \|] = Y$. Further, we may also assume that t_1 is the first such time in the interval I. Hence by hypothesis (i) and Lemma 2 we have

$$E[V(t, z(t; \omega); \omega)] \leqslant r(t, t_0, r(t_0)) + U(t; \omega) \text{ for all } t \geqslant t_0.$$

In view of (12) and (13) it follows that

$$V_m^Y(t_1) \leqslant E[V(t_1, z(t_1; \omega); \omega)] \leqslant r(t_1, t_0, r(t_0)) + U(t_1; \omega) < V_m^Y(t_1). \quad \text{...(13)}$$

This contradicts the existence of $t_1 \in I$ with $E[\| z(t_1; \omega) \|] = Y$ which completes the proof of the theorem.

Our next theorem shows that under some suitable conditions on the functions V and g the stochastic control system (3) will be finitely uniformly stable in the mean.

Theorem 2 — The stochastic control system (3) is finitely uniformly stable in the mean with respect to $(\alpha, Y, t_0, T, \| \cdot \|)$, $\alpha < Y$ if there exist functions V and g such that

(i) For all $t \in I$, and $z \in \bar{B}(Y) - B(\alpha)$, (7) and (8) hold, and

(ii) the maximal solution $r(t, t_1, r(t_1))$ of (11) with initial condition $r(t_1) = V_M^{\alpha}(t_1) - U(t_1; \omega)$ at any $t_1 \in I$ satisfies the condition

$$r(t, t_1, r(t_1)) < V_m^Y(t) - U(t; \omega) \text{ for all } t > t_1, t \in I. \quad \text{...(14)}$$

PROOF : Let $z(t; \omega)$ be any arbitrary trajectory of (3) in the mean with $\| z(t_0; \omega) \| < \alpha$. Let there exist a $t_2 \in I$ such that $E[\|z(t_2; \omega) \|] = Y$. As in the Theorem 1 it may be assumed that t_2 is the first such time in the interval I. Then there exists a $t_1 \in I$, $t_0 < t_1 < t_2$ such that

$$E[\| z(t_1; \omega) \|] = \alpha \text{ and } Y \geqslant E[\| z(t; \omega) \|] \geqslant \alpha$$

for $t \in [t_1, t_2]$. Hence by hypothesis (i) and Lemma 2 we have

$$E[V(t, z(t; \omega); \omega)] \leqslant r(t, t_1, r(t_1)) + U(t; \omega) \quad \text{...(15)}$$

for all $t \in [t_1, t_2]$.

Therefore in view of (14) and (15), it follows that

$$V_m^Y(t_2) \leqslant E[V(t_2, z(t_2; \omega); \omega)] \leqslant r(t_2, t_1, r(t_1)) + U(t_2; \omega) < V_m^Y(t_2).$$

This contradicts the existence of $t_2 \in I$ with $E[\| z(t_2; \omega) \|] = Y$. We then conclude that

$$E[\| z(t; \omega) \|] < Y \text{ for all } t \in I.$$

This argument is independent of the particular trajectory chosen, hence it will hold for all trajectories $z(t; \omega)$ with $\| z(t_0; \omega) \| < \alpha$. This completes the proof of the theorem.

Theorem 3 below establishes the finite quasi-expansive stability in the mean of the stochastic control system (3).

Theorem 3 — The stochastic control system (3) is quasi-expansively finitely stable in the mean with respect to $(\alpha, \beta, t_0, T, \| \cdot \|)$, if there exist functions V and g such that

(i) for all $t \in I$, $z(t; \omega) \notin B(\beta)$, (7) and (8) hold; and

(ii) the maximal solution $r(t, t_1, r(t_1))$ of (11) with initial condition $r(t_1) = V_M^{\beta}(t_1) - U(t_1; \omega)$ at any $t_1 \in I$ satisfying the condition

$$r(t_0 + T, t_1, r(t_1)) < V_m^{\beta}(t_0 + T) - U(t_0 + T; \omega); \qquad ...(16)$$

(iii) for all $z \notin B(\beta)$

$$E[V(t_0 + T, z(t; \omega); \omega)] \geqslant V_m^{\beta}(t_0 + T). \qquad ...(17)$$

Proof : To prove this theorem it will be enough to show that for every trajectory of (3), the condition $E[\| z(t_0; \omega) \|] < \alpha$ implies $E[\| z(t_0 + T; \omega) \|] < \beta$ let there exist a trajectory $z(t; \omega)$ of (3) with $\| z(t_0; \omega) \| < \alpha$ such that $E[\| z(t_1; \omega) \|] = \beta$ and $E[\| z(t; \omega) \|] \geqslant \beta$ for all $t \in [t_1, t_0 + T)$, $t_1 \in I$. Hence by the condition (i) of the theorem and by Lemma 2 we have

$$E[V(t, z(t; \omega); \omega)] \leqslant r(t, t_1, r(t_1)) + U(t; \omega) \qquad ...(18)$$

for all $t \in [t_1, t_0 + T]$. Therefore, from the inequalities (16), (17) and (18), we obtain

$$\begin{aligned} V_m^{\beta}(t_0 + T) &\leqslant E[V(t_0 + T, z(t_0 + T; \omega); \omega] \\ &\leqslant r(t_0 + T, t_1, r(t_1)) + U(t_0 + T; \omega) \\ &< V_m^{\beta}(t_0 + T) \end{aligned}$$

which is a contradiction. Hence the theorem.

Finally we state and prove the following theorem which yields sufficient conditions for finite stability in the mean in the sense of the Definitions 3 and 5.

Theorem 4 — The stochastic control system (3) is quasi-contractively finitely stable in the mean with respect to $(\alpha, \beta, t_0, T, \|\cdot\|)$ if there exist functions V and g such that

(i) for all $t \in I$ and $z \notin B(\beta)$, (7) and (8) hold; and

(ii) the maximal solution $r(t, t_1, r(t_1))$ of (11) with initial condition $r(t_1) = V_m^\beta(t_1) - U(t_1; \omega)$ at any $t_1 \in I$, satisfying the condition

(a) $r(t_0 + T, t_1, r(t_1)) < V_m^\beta(t_0 + T) - U(t_0 + T; \omega)$;

(iii) for all $z \notin B(\beta)$

$$E\,[V(t_0 + T, z(t_0 + T; \omega); \omega] \geqslant V_m^\beta(t_0 + T).$$

Proof : Let there exist a trajectory $z(t; \omega)$ of (3) with $\| z(t_0; \omega) \| < \beta$ such that $E\,[\|z(t_0 + T; \omega)\|] \geqslant \beta$, then there exists a $t_1 \in I$ such that

$$E\,[\| z(t_1; \omega) \|] = \beta \text{ and } E\,[\| z(t; \omega) \|] > \beta$$

for all $t \in [t_1, t_0 + T]$, $t_1 > t_0$. In this case the condition (i) holds good and by applying Lemma 2, we have

$$E\,[V(t, z(t; \omega); \omega)] \leqslant r(t, t_1, r(t_1)) + U(t; \omega).$$

Therefore by virtue of (iia) and (iii) we have

$$\begin{aligned} V_m^\beta(t_0 + T) &\leqslant E\,[V(t_0 + T, z(t_0 + T; \omega); \omega)] \\ &\leqslant r(t_0 + T, t_1, r(t_1)) + U(t_0 + T; \omega) \\ &< V_m^\beta(t_0 + T) \end{aligned}$$

which is a contradiction and hence $E\,[\| z(t_0 + T; \omega) \|] < \beta$. This completes the proof of the theorem.

Remark : By applying the Lemmas used herein and the theorems, one can easily formulate also the theorems of stochastic expansive and contractive finite stabilities in the mean.

Acknowledgement

The author is much indebted to Dr B. G. Pachpatte, Professor and Head, Department of Mathematics and Statistics, Marathwada University, Aurangabad, for suggesting this problem and for making useful suggestions during the preparation of this paper.

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