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THE TIME DEPENDENT POISSON QUEUE AND MOVING BOUNDARY PROBLEMS FOR THE HEAT AND WAVE EQUATION

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In the classical analysis of the melting of a semi-infinite one-dimensional slab of ice (Stefan problem) due to heating at one end, there results a partial differential equation with a moving boundary. A similar situation arises in the study of a vibrating semi-infinite string with a moving end point and in the study of a single-server queue with time-dependent Poisson arrival and service of items. The analysis of these problems leads to integral transforms which are generalizations of the Laplace transform. A method of inverting these transforms is obtained. In particular there is developed a new derivation of the classical results concerning the stationary Poisson single-server queue.

1. INTRODUCTION

The classical Stefan problem (Stefan 1890, Rubinstein 1948, Datzeff 1950, Kolodner 1956), concerned with the melting of a linear, semi-infinite, one-dimensional bar of ice leads to a motion of the solid-liquid interface proportional to the square root of time.

This is a free boundary problem where the motion of the boundary is determined by physical events at the interface of the two phases. If on the other hand the motion of the boundary of the semi-infinite bar as well as initial and boundary conditions are prescribed, and a solution of the equation of heat conduction satisfying these requirements is called for, then the literature contains methods for finding that solution only for a few isolated cases, such as linear t-motion (Carslaw and Jaeger 1959), the above-mentioned Stefan $t^{1/2}$-motion (Datzeff 1950, Kolodner 1956, Redozubov 1957) and some specific cases for $t^{3/2}$-motion (Redozubov 1957) of the boundary.

In section 2 we shall give a method of solution of this problem for any type of motion of the boundary, limited only by the condition that the velocity of the moving boundary be bounded.

This method is also applicable to other geometries, higher dimensions and other differential equations, and in section 3 will be illustrated by a problem involving the scalar one-dimensional wave equation.

The transient behavior of the infinite, single channel queue with time-dependent Poisson-distributed arrival and service rates yields to an analysis by a discrete variant of our method, and is discussed in section 4.

We are concerned with presenting a method of obtaining the solutions to these problems. The existence of these solutions under various general assumptions has been treated mostly by iterative processes (Chiang 1965, Dronkers 1949).

2. Heat Conduction on the Semi-infinite Line with Moving Boundary

We desire to find a function $u(x, t)$ satisfying

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \text{ for } g(t) < x < \infty,\ 0 < t < \infty$$

with proper behavior for large x and t, and with

$$u(x, 0) = u_0(x) \quad \text{for } 0 < x < \infty$$

$$u(g[t], t) = f(t) \quad \text{for } 0 < t < \infty$$

where $g(0) = 0$, $|f(t)|$ is bounded and $u_0(x)$, $f(t)$ and $g(t)$ are such that all analytical steps that follow are justified; in addition, we assume that $|g'(t)|$ is bounded for all $t \geqslant 0$ with the possible exception of the neighbourhoods of a finite number of points.

The substitution $y = x - g(t)$ leads to $v(y, t) \equiv u(x, t)$ and the equation

$$\frac{\partial v}{\partial t} = g'(t)\frac{\partial v}{\partial y} + \frac{\partial^2 v}{\partial y^2}, \quad 0 < y < \infty, \quad 0 < t < \infty \qquad \text{...(2.1)}$$

with the conditions

$$v(y, 0) = u_0(y), \quad 0 < y < \infty$$

$$v(0, t) = f(t), \quad 0 < t < \infty.$$

We note in passing that eqn. (2.1) describes the diffusion of charged particles on the semi-infinite line under the influence of a time-dependent electric field.

A Laplace transformation with respect to y leads from eqn. (2.1) to

$$\frac{\partial \bar{v}}{\partial t} = g'(t)\,[p\bar{v} - f(t)] + [p^2\bar{v} - pf(t) + k(t)] \qquad \text{...(2.2)}$$

where

$$k(t) = -\left.\frac{\partial v(y, t)}{\partial y}\right|_{y=0} \equiv -\frac{\partial v(0, t)}{\partial y}$$

and

$$\tilde{v}(p, t) = \int_0^\infty v(y, t) \exp[-py]\, dy.$$

The Laplace transform will be indicated by a tilde and unless there is ambiguity, no additional reference will be made to the variable affected.

From eqn. (2.2) we get at once

$$\tilde{v}(p, t) = \exp[pg(t) + pt^2] \{ \int_0^t [k(r) - g'(r) f(r) - pf(r)]$$

$$\exp[-pg(r) - p^2 r]\, dr + \tilde{u}_0(p)\} \qquad \text{...(2.3)}$$

Since a solution $v(y, t)$ must remain finite as t becomes infinite, we conclude from eqn. (2.3) that for p in a region $\mathcal{S}$ covering a semi-infinite part of the positive real axis :

$$\int_0^\infty [k(r) - g'(r) f(r) - pf(r)] . \exp[-pg(r) - p^2 r]\, dr = -\tilde{u}_0(p)$$

and hence after integration by parts that for $p \in \mathcal{S}$:

$$\int_0^\infty k(r) . \exp[-pg(r) - p^2 r]\, dr = f(0)\, p^{-1} - \tilde{u}_0(p)$$

$$+ p^{-1} . \int_0^\infty f'(r) \exp[-pg(r) - p^2 r]\, dr. \qquad \text{...(2.4)}$$

A method of solving eqn. (2.4) for $k(t)$ will be given in Section 5. Substitution into eqn. (2.3) and inversion of the resulting Laplace-transform gives the solution to our problem.

We point out that eqn. (2.4) with the function $g(t)$ as the unknown also serves in describing the unknown motion of the boundary in the related free boundary problem. Here $k(t)$, the flux, is determined by physical events at the phase-interface.

3. The Wave Equation for a Semi-infinite String with a Moving End Point

We are concerned with finding a function $u(x, t)$ which satisfies

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \text{ for } g(t) < x < \infty,\ 0 < t < \infty$$

with proper behaviour for large x and t, and with

$$u(x, 0) = u_0(x),\ \left.\frac{\partial u(x, t)}{\partial t}\right|_{t=0} \equiv \frac{\partial u(x, 0)}{\partial t} = u_1(x) \text{ for } 0 < x < \infty$$

and

$$u(g(t), t) = f(t) \text{ for } 0 < t < \infty$$

where $g(0) = 0$ and $| f(t) |$ is bounded. We assume that $u_0(x)$, $u_1(x)$, $f(t)$ and $g(t)$ are such that all analytical steps that follow are justified. We furthermore suppose that $| g'(t) | < 1$, i.e., the speed at which the boundary moves is less than the wave velocity.

Proceeding as in section 2 we first put $y = x - g(t)$ and $v(y, t) \equiv u(x, t)$. Then we Laplace-transform with respect to y and obtain a differential equation for $\tilde{v}(p, t)$ which after letting

$$\tilde{v}(p, t) = f(t)\, p^{-1} + \tilde{w}(p, t) \exp[pg(t)] \quad \text{and} \quad k(t) = -\frac{\partial u(0, t)}{\partial y}$$

leads to

$$\frac{\partial^2 \tilde{w}}{\partial t^2} - p^2 \tilde{w} = [k(t) . (1 - \{g'(t)\}^2) - f''(t)\, p^{-1}] . \exp[-pg(t)]$$

with the solution

$$\tilde{w}(p, t) = \exp[-pt](-2p)^{-1} . \left\{ \int_0^t \exp(2pr)\, R(p, r)\, dr + A \right\}$$

$$+ \exp[pt](2p)^{-1} . \left\{ \int_0^t R(p, r)\, dr + B \right\}$$

where

$$R(p, r) = \exp[-pr - pg(r)] . [k(r) . (1 - \{g'(r)\}^2) - f''(r)\, p^{-1}]$$

$$A = [\tilde{u}_1(p) - f'(0)\, p^{-1}] - p . [g'(0) + 1] . [\tilde{u}_0(p) - f(0)\, p^{-1}]$$

and

$$B = [\tilde{u}_1(p) - f'(0)\, p^{-1}] - p . [g'(0) - 1] . [\tilde{u}_0(p) - f(0)\, p^{-1}].$$

Since a solution must remain finite as $t \to \infty$, we conclude from the preceding expression for $\tilde{w}$ that for p in a region $\mathcal{S}$ (here actually a half plane) covering a semi-infinite part of the positive real axis :

$$\int_0^\infty R(p, r)\, dr = -B \qquad ...(3.1)$$

from which $k(t)$ can be obtained by the method discussed in section 5.

4. The Time-Dependent Poisson-Queue

We shall obtain the transient behaviour of the single-channel queue with time-dependent Poisson-distributed arrival and service rates.

If $\lambda(t)\,\triangle$ is the probability for an arrival, and $\mu(t)\,\triangle$ is the probability for a service between times t and $t+\triangle$, with the probabilities for multiple arrivals and services being $o(\triangle)$, and if $P_n(t)$ denotes the probability of there being n items in the queue (including any item which may be in service), then one has the system of equations (Saaty 1961)

$$P'_0(t) = -\lambda(t)\,P_0(t) + \mu(t)\,P_1(t)$$

$$P'_n(t) = -[\lambda(t)+\mu(t)]\,P_n(t) + \lambda(t)\,P_{n-1}(t) + \mu(t)\,P_{n+1}(t) \quad \text{for} \quad n \geqslant 1, \qquad \ldots(4.1)$$

with the conditions that $P_n(t) \geqslant 0$ for $n \geqslant 0$, $\Sigma_{n=0}^{\infty} P_n(t) = 1$ and an initial state condition which on account of the linearity of the system may be taken to be $P_n(0) = \delta(i, n)$ where $\delta(i, n)$ is the usual Kronecker delta which vanishes for all n except $n = i$ where its value is 1.

Introduction of the generating function

$$f(z, t) = \sum_{n=0}^{\infty} P_n(t)\, z^n$$

leads to the equation

$$\frac{\partial f}{\partial t} = \{-[\lambda(t)+\mu(t)] + \lambda(t)\,z + \mu(t)\,z^{-1}\}\,f + (1-z^{-1})\,\mu(t)\,P_0(t)$$

and hence to

$$f(z, t) = \exp[L(t)\,(z-1) + M(t)\,(z^{-1}-1)]$$
$$\times\left[(1-z^{-1})\int_0^t \exp[L(r)\,(1-z) + M(r)\,(1-z^{-1})]\,.\,P_0(r)\,\mu(r)\,dr + z^i\right] \qquad \ldots(4.2)$$

where

$$L(t) = \int_0^t \lambda(r)\,dr \quad \text{and} \quad M(t) = \int_0^t \mu(r)\,dr.$$

We shall assume that $L(\infty) = M(\infty) = \infty$, i.e., that arrivals and services occur eventually with probability one.

Since as $t \to \infty$, $f(z, t)$ is to remain finite for $|z| \leqslant 1$, we conclude from eqn. (4.2) that

$$\int_0^{\infty} P_0(r)\,\mu(r)\exp[L(r)\,(1-z) + M(r)\,(1-z^{-1})]\,dr = z^{1+i}(1-z)^{-1}$$

for z in a region S_1 covering a portion of the positive real axis starting at the origin.

Putting $y = z^{-1} - 1$ there results then

$$\int_0^\infty P_0(r)\,\mu(r)\exp\left[L(r)\,y(1+y)^{-1} - M(r)\,y\right] dr = y^{-1}(1+y)^{-1} \qquad \ldots(4.3)$$

for y in a region $\mathcal{S}_2$ covering a semi-infinite part of the positive real axis.

The method given in section 5 enables one to obtain $P_0(t)$ from eqn. (4.3).

The moments $M_1(t) = \Sigma_{n=0}^{\infty}\, nP_n(t)$ and $M_2(t) = \Sigma_{n=0}^{\infty}\, n^2P_n(t)$ can be seen either directly from eqns. (4.1), or by evaluating $\frac{\partial f}{\partial z}$ and $\frac{\partial^2 f}{\partial z^2}$ at $z = 1$, to be given by

$$M_1(t) = L(t) - M(t) + \int_0^t P_0(r)\,\mu(r)\,dr$$

and

$$M_2(t) = L(t) + M(t) - \int_0^t P_0(r)\,\mu(r)\,dr + 2\int_0^t [\lambda(r) - \mu(r)]\,.\,M_1(r)\,dr.$$

Putting

$$A(y, r) = [-\lambda(r)y(1+y)^{-1} + \mu(r)\,y]\exp\left[L(r)\,y(1+y)^{-1} - M(r)y\right]$$

we note, inverting orders of integration and making use of eqn. (4.3), that

$$\int_0^\infty\int_0^t P_0(r)\,\mu(r)\,dr\,.\,A(y, t)\,dt$$

$$= \int_0^\infty P_0(r)\,\mu(r)\exp\left[L(r)\,y(1+y)^{-1} - M(r)y\right] dr = y^{-1}(1+y)^{-i} \qquad \ldots(4.4)$$

for $y \in \mathcal{S}_2$.

Thus $M_1(t)$ can be obtained directy by use of the method of Section 5 without first obtaining $P_0(t)$. Similar results hold for the higher moments.

If $\lambda(t)$ and $\mu(t)$ are constant, eqn. (4.3) becomes, after putting $s = -\lambda y(1+y)^{-1}+\mu y$,

$$\int_0^\infty P_0(r)\exp(-sr)\,dr = \alpha^{i+1}\mu^{-1}(1-\alpha)^{-1}$$

where

$$\alpha = \{\lambda + \mu + s - [(\lambda + \mu + s)^2 - 4\lambda\mu]^{1/2}\}/2\lambda.$$

Thus we have derived the classical result for the stationary Poisson-single-channel-Erlang queue [Saaty 1961, (4–9)] without the usual use of the analyticity condition.

We conclude this section by mentioning the publications of Luchak (1956), Clarke (1956) and Keilson (1962) concerned with the time-dependent Poisson-Queue. Luchak uses matrix expansions while the other two authors reduce the problem to the solution of Volterra integral equations, Clarke via a telegraphy partial differential equation, and Keilson via his powerful Green's functions method.

5. The Inversion of the Integral Transforms

The integral eqns. (2.4), (3.1), (4.3) and (4.4) represent generalizations of the Laplace transform to which they reduce for the special situations where $g(t) = at$, a being any constant, and $\lambda(t)$ and $\mu(t)$ being constants. These integral transforms may be inverted under fairly general conditions by extending a method which Erdélyi (1943) (see also Ribaric 1959) gave for Laplace transforms. Equation (3.1) is reducible to an ordinary Laplace transform.

We illustrate our method by considering eqn. (2.4). Equations (3.1), (4.3) and (4.4) are treated quite similarly. (cf. Appendix B)

We are concerned with the equation

$$\int_0^\infty k(r) \,.\, \exp\left[-pg(r) - p^2 r\right] dr = h(p), \quad p \in \mathcal{S} \qquad \ldots(5.1)$$

where $\mathcal{S}$ covers a semi-infinite part of the positive real axis.

We rewrite (5.1) in the form

$$\int_0^\infty [r^{1/4} \,.\, k(r) \exp(-cr)] \,.\, \{r^{-1/4} \exp[-pg(r) - p^2 r + cr]\}\, dr = h(p)$$

where c is chosen such that $r^{1/4} \,.\, k(r) \,.\, \exp(-cr) \in L_2(0, \infty)$. The nature of the boundary conditions imposed by us will insure the existence of the solution $k(r)$ with the necessary properties. The factor $r^{1/4}$ will insure the necessary square-integrability at the origin and is introduced to compensate for a $t^{-1/2}$-behavior of $k(t)$ at $t = 0$. Such behaviour occurs and such a factor is necessary at every point of discontinuity of the given boundary temperature. What follows is readily seen to be extendable to handle a finite number of such discontinuities. The function $g(x)$ will always be assumed to have the properties mentioned earlier and restated in the lemma below. In the following text we assume that $g'(x)$ is bounded in a neighbourhood of $x = 0$. This restriction is removed in Appendix A.

We shall now prove the closure, and hence completeness, in $L_2(0, \infty)$ of the set of functions $\{v_n(x)\} = x^{-1/4}.\{\exp[-\lambda_n^{1/2} g(x) - \lambda_n x + cx]\}$, where $\lambda_n > 0$, $n = 1, 2, \ldots$, is a monotone sequence of distinct real numbers which are large enough so that the $v_n(x)$ belong to $L_2(0, \infty)$, $\lambda_n^{1/2} \in \mathcal{S}$ and $\sum_{n=1}^{\infty} \lambda_n^{-1-\epsilon} = \infty$, for some $\epsilon > 0$. Furthermore, if for any $a > 0$ there exists a $M(a) > 0$ such that $g(x) > M(a)$ for $x > a$, then we may take $\epsilon = 0$.

Consider the function

$$F(z) = \int_0^\infty x^{-1/4} . \exp[-z^{1/2}g(x) - zx + cx] . f(x)\, dx$$

which for any $f(x) \in L_2(0, \infty)$ is bounded and analytic in a right sector, $\mathcal{R}$, of angle $\pi - \alpha$, $\alpha > 0$. We note that $\mathcal{R}$ is a closed right-half plane in case for any $a > 0$ there exists a $M(a) > 0$ such that $g(x) > M(a)$ for $x > a$. (cf. Appendix A). [Schwarz inequality and (Titchmarsh 1939, p. 100 and p. 397, ex. 17).] From $F(\lambda_n) = 0$, $n = 1, 2, \ldots$, and the divergence of the series given above, one concludes then by Carleman's theorem (Titchmarsh 1939, p. 131) that $F(z) \equiv 0$. In case $\mathcal{R}$ is not a full half-plane but only a right sector, one first maps conformally by $z^* = z^{1+\epsilon}$, $\epsilon > 0$, before applying Carleman's theorem.

The closure of the set $\{v_n(x)\}$ follows from the following:

Lemma — If

$$F(z) = \int_0^\infty x^{-1/4} . \exp[-z^{1/2}g(x) - zx + cx] . f(x)\, dx = 0 \qquad \ldots(5.2)$$

for all real z larger than a given z_0, and if $g(x) \in C(0, \infty)$, $g(0) = 0$ and $|g'(x)|$ is bounded for $x \geqslant 0$ with the possible exception of the neighbourhoods of a finite number of points different from the origin, then if $f(x) \in L_2(0, \infty)$, $f(x)$ vanishes almost everywhere. (Note: The exceptional points for $g'(x)$ allow for a t^α-type motion, $0 < \alpha < 1$, of the boundary.)

PROOF: Introducing the bounded and continuous function

$$f_1(x) = \int_0^x y^{-1/4} \exp[-a^{1/2}g(y) - ay + cy] f(y)\, dy \qquad \ldots(5.3)$$

where $a > z_0$, we have for $z > a$ that

$$F(z) = \int_0^\infty \exp[(a^{1/2} - z^{1/2})\, g(x) + (a - z)\, x]\, df_1(x)$$

(*equation continued on p.* 575)

$$= -(a^{1/2} - z^{1/2}) \int_0^\infty [g'(x) + a^{1/2} + z^{1/2}] \exp [(a^{1/2} - z^{1/2}) . g(x) + (a - z)x] . f_1(x)\, dx = 0.$$

Thus the problem has been reduced to showing that $f_1(x)$ vanishes identically, for this will at once imply in virtue of eqn. (5.3) that $f(x) = 0$ almost everywhere.

We may write

$$\int_0^\infty K(x, z) . f_1(x)\, dx \equiv 0, \quad z \in \mathcal{R}$$

$\mathcal{R}$ being a region of analyticity containing a semi-infinite portion of the real z-axis, with $K(x, z) = [g'(x) + a^{1/2} + z^{1/2}]\, e^{-wx}$ where $w = A(x)\,(z^{1/2} - a^{1/2}) + z - a$ and $| A(x) |$ is bounded.

On account of analyticity mentioned earlier we also have for all n

$$\int_0^\infty \frac{\partial^n K(x, z)}{\partial z^n} . f_1(x)\, dx = 0, \quad z \in \mathcal{R}. \qquad \text{...(5.4)}$$

Proceeding by indirect proof we shall show that eqn. (5.4) is contradicted for n large enough if $f_1(x_0) \neq 0$ for some $x_0 > 0$.

We readily find that $\partial^n K/\partial z^n = K(x, z) . [(-x)^n + \epsilon(x, z)]$, where $\epsilon(x, z) = 0(x^n)$ uniformly for $z \geqslant z_1 > 0$ as $x \to \infty$, and where $\epsilon(x, z)$ for $0 \leqslant x < x_1$ is bounded and $\to 0$ uniformly in x as $z \to \infty$.

Letting $z = a + n/x_0$ we note on account of the boundedness of $g'(x)$ that $\partial^n K/\partial z^n$ equals $(xe^{-x/x_0})^n$ except for factors which are unessential as far as the reasoning that follows is concerned.

Consider the integral $I(a, b) = \int_a^b x^n e^{-nx/x_0}\, dx$.

We have

$$I(x_0 - (x_0/m),\ x_0 + (x_0/m)) = (x_0)^{n+1} \int_{(m-1)/m}^{(m+1)/m} y^n e^{-ny}\, dy$$

$$\geqslant (x_0)^{n+1} . (2/m) . (1 - (1/m))^n . e^{-n(1-(1/m))}, \text{ where } m = n^{1/4}$$

while

$$I(0, \infty) = (x_0)^{n+1} . n!/n^{n+1} < (x_0)^{n+1} . (2\pi/n)^{1/2} . e^{-n} . \left(1 + \frac{1}{12n - 1}\right).$$

Hence we find for n large enough that

$$\frac{I(x_0-(x_0/m),\, x_0+(x_0/m))}{I(0,\infty)} > (2/\pi)^{1/2} \,.\, m. \qquad ...(5.5)$$

The continuity of $f_1(x)$ at x_0 implies that $f_1(x)$ is one sign and absolutely above a certain value in a neighbourhood of x_0. Since $f_1(x)$ is also bounded, we conclude from the inequality (5.5) that the contribution of the range

$$x_0 - (x_0/m) < x < x_0 + (x_0/m)$$

to the integral in eqn. (5.4) cannot be cancelled by the rest of the integral, and hence that eqn. (5.4) fails for n large enough.

Having established the completeness of the set $\{v_n(x)\}$ we pass by the Schmidt process to the complete orthonormal set $\{w_n(x)\}$:

$$w_n(x) = \sum_{m=1}^{n} c_{nm} v_m(x).$$

Expansion of $x^{1/4} \,.\, k(x) \,.\, \exp(-cx)$ leads to

$$x^{1/4} \,.\, k(x) \exp(-cx) = \underset{N\to\infty}{\text{l.i.m.}} \sum_{n=1}^{N} w_n(x) \int_0^\infty y^{1/4} k(y) \exp(-cy)\, w_n(y)\, dy$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \sum_{n=1}^{N} w_n(x) \sum_{m=1}^{n} c_{nm} h(\lambda_m^{1/2}) \qquad ...(5.6)$$

and hence to a solution of eqn. (5.1).

Since our Lemma implies that the inversion of eqn. (5.1) is unique (except on a set of measure zero) within the class of functions under consideration and since existence theorems guarantee solutions belonging to that class, we conclude that eqn. (5.6) yields the desired solution of eqn. (2.4).

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APPENDIX A

The Boundedness of the Function $F(z)$ of Section 5

Recall that $g(0) = 0$, $g(x)$ is continuous for $x \geqslant 0$, $g'(x)$ is bounded for $x \geqslant 0$ with the possible exception of the neighbourhoods of a finite number of points.

Case 1 — Let us consider first the situation where the origin is not one of these exceptional points. There exists in that instance a B such that $|\, g(x)\, | < Bx$ for $x \geqslant 0$.

Application of the Schwarz inequality to the expression representing $F(z)$ in Section 5 leads us to consider the integral

$$I(z) = \int_0^\infty x^{-1/2} \exp\left[-\, 2g(x)\, z^{1/2} - 2zx + 2cx\right] dx$$

the boundedness of which we shall now investigate.

Putting $z = re^{i\theta}$, we note that $|\, I\, | < \int_0^\infty x^{-1/2} \exp\left[2x\,.\,(Br^{1/2}\cos(\theta/2) - r\cos\theta + c)\right] dx$ which is uniformly bounded for all r larger than some r_1 provided $-(\pi/2) + \alpha \leqslant \theta \leqslant (\pi/2) - \alpha$, $\alpha > 0$.

Thus we have established the boundedness of $I(z)$ in a closed right-sector : $-(\pi/2) + \alpha \leqslant \theta \leqslant (\pi/2) - \alpha$, $\alpha > 0$; i.e., in a sector of sectorial angle arbitrarily close to π.

In case $g(x)$ is such that for any $a > 0$ there exists a $M(a) > 0$ such that $g(x) > M(a)$ for $x > a$, then we write $I(z) = I_1(z) + I_2(z)$, where I_1 and I_2 are the integrals over the ranges (0, 1) and $(1, \infty)$, respectively. We have $| I_1 | < \int_0^1 2e^{2cy^2}dy$, while $| I_2 | < \int_1^\infty x^{-1/2} \exp [- 2M(1)r^{1/2} \cos (\theta/2) - 2xr \cos \theta + 2cx] \, dx$ which is uniformly bounded for all r larger than some r_1, provided that $-(\pi/2) \leqslant \theta \leqslant (\pi/2)$. In this instance the integral $I(z)$ is therefore bounded in a closed half plane.

Case 2 — If $g'(x)$ is unbounded in every neighbourhood of $x = 0$, as for example if $g(x) = x^{1/2}$, then we replace the integral eqn. (5.1) by

$$\int_0^\infty k(r) \,.\, \exp [- bp - pg(r) - p^2 r] \, dr = h(p) \,.\, e^{-bp}, \; p \in \mathcal{S}, \text{ where } b > 0.$$

We are now lead to consider the integral

$$I(z) = \int_0^\infty x^{-1/2} \exp [- 2bz^{1/2} - 2g(x) \, z^{1/2} - 2zx + 2cx] \, dx$$

which is readily seen to be uniformly bounded in a right sector.

The remainder of the argument in this instance is quite analogous to that given in Section 5.

(The corresponding analysis for the wave equation and queueing examples is somewhat simpler.)

Appendix B

Part I : The Wave Equation Example

Equation (3.1) is of the form $\int_0^\infty \exp [- pg(r) - pr] \, k^*(r) \, dr = h(p)$ for p in a region $\mathcal{S}$ which covers a semi-infinite part of the positive real axis.

Our boundary conditions permit us to assume that $k^*(r) \exp [- cr] \in L_2(0, \infty)$, $c > 0$, and we rewrite the integral equation with $k^*(r) \,.\, \exp [- cr]$ as our unknown function : $\int_0^\infty k^*(r) \exp [-cr] \,.\, \exp [- pg(r) - pr + cr] \, dr = h(p)$.

We proceed quite similarly as in our discussion of the Heat-Equation example. We introduce the set of functions $\{\nu_n(x)\} = \exp [- \lambda_n g(x) - \lambda_n x + cx]$ where the

$\lambda_n > 0$, $n = 1, 2, \ldots$, are large enough so that $v_n(x) \in L_2(0, \infty)$, $\lambda_n \in \mathcal{S}$ and $\sum_{n=1}^{\infty} \lambda_n^{-1-\epsilon} = \infty$ for some $\epsilon > 0$.

The completeness argument proceeds by Carleman's theorem via the function $F(z) = \int_0^\infty \exp[-zg(x) - zx + cx] \,.\, f(x)\, dx$ which is bounded and analytic in a closed right-sector of angle $\pi - \alpha$, $\alpha > 0$. From $F(z) = 0$, $z > z_0$, we get, after introducing $f_1(x) = \int_0^x \exp[-ag(y) - ay + cy] \,.\, f(y)\, dy$, $a > z_0$, that

$$\int_0^\infty [g'(x) + 1] \,.\, \exp[g(x)(a - z) + x(a - z)] \,.\, f_1(x)\, dx = 0 \text{ for } z > a.$$

We recall that $| g(x) | < x$ and have

$$K(x, z) = [g'(x) + 1] \exp[g(x)(a - z) + x(a - z)]$$
$$= [g'(x) + 1] \exp[(A + 1)\, x(a - z)], \quad | A | < 1$$

with the consequence that for

$$z = a + n/[x_0(A + 1)] : \partial^n K(x, z)/\partial z^n$$
$$= [g'(x) + 1](A + 1)^n (-1)^n (xe^{-x/x_0})^n.$$

The remainder of the argument is identical to that in section 5, except that instead of $h(\lambda_m^{1/2})$ we have $h(\lambda_m)$ in the equation corresponding to eqn. (5.6).

It is of interest to note that the substitution $g(r) + r = s$ reduces eqn. (3.1) to an ordinary Laplace transform equation. Such a reduction is not possible in the heat flow and queueing instances.

Part II : The Queueing Example

If in eqn. (4.3) we put $u = M(r)$, $\rho(u) = \lambda(r)/\mu(r)$, $P_0^*(u) = P_0(M^{-1}(u))$ and $L(r) = \int_0^r \lambda(x)\, dx = \int_0^u \rho(s)\, ds \equiv R(u)$ we obtain the integral equation

$$\int_0^\infty P_0^*(u) \,.\, \exp[R(u)\, y(1 + y)^{-1} - uy]\, du = y^{-1}(1 + y)^{-i}$$

for y in a region $\mathcal{S}_2$ which covers a semi-infinite part of the positive real axis.

It is quite reasonable to assume that $\lambda(r)$ and $\mu(r)$ are such that $R(u) = 0(u)$ as $u \to \infty$ and that $P_0^*(u)\, e^{-cu} \in L_2(0, \infty)$ for some $c > 0$. We furthermore assume that $\rho(u)$ is bounded, but the analysis can be extended to handle the situation where $\rho(u)$ becomes unbounded in the neighbourhoods of a finite number of points.

Proceeding as in the earlier instances, we introduce the set of functions $\{\nu_n(x)\} = \exp[R(x) . \lambda_n(1 + \lambda_n)^{-1} - x\lambda_n + cx]$ where the $\lambda_n > 0$, $n = 1, 2, \ldots$, are large enough so that $\nu_n(x) \in L_2(0, \infty)$, $\lambda_n \in \mathcal{S}_2$ and $\sum_{n=1}^{\infty} \lambda_n^{-1-\epsilon} = \infty$ for some $\epsilon > 0$. Here

$$F(z) = \int_0^{\infty} \exp[R(x)(z(1+z)^{-1} - a(1+a)^{-1}) - x.(z-a)] . df_1(x) = 0$$

for

$$z > a > z_0, \text{ with } f_1(x) = \int_0^x \exp[a(1+a)^{-1} R(y) - ay + cy] . f(y)\, dy.$$

We are thus lead to .

$$\int_0^{\infty} K(x, z) . f_1(x)\, dx = 0, \quad \text{for } z > a > z_0,$$

where

$$K(x, z) = [R'(x) . (1+z)^{-1} . (1+a)^{-1} - 1] . e^{-wx}$$

with $w = (z - a) . [1 - A(x) . (1+z)^{-1} . (1+a)^{-1}]$

and bounded $|A(x)|$.

We readily find that if $z = a + n/x_0$, then $\partial^n K/\partial z^n$ equals $(xe^{-x/x_0})^n$ except for unimportant factors. The remainder of the argument parallels that of the earlier examples.