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CERTAIN FRACTIONAL q-INTEGRAL OPERATORS

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In this paper we introduce two operators K_q and I_q of fractional q-integration which may be regarded as extensions of Upadhyay's (1970) operators. These operators along with M_q and L_q operators, the q-analogues of Mellin and Laplace transforms respectively, give some new results. Certain other interesting theorems are also derived.

1. INTRODUCTION

AlSalam (1966) defined a fractional q-integral operator through the q-integral

$$K_q^{\eta,\alpha} f(x) = \frac{q^{-\eta}x^{\eta}}{(1-q)^{1-\alpha}} \Pi_q \begin{bmatrix} \alpha; \\ 1 \end{bmatrix} \int_x^{\infty} [y - x]_{\alpha-1}\, y^{-\eta-\alpha} f(yq^{1-\alpha})\, d(y; q) \quad \ldots(1.1)$$

where $\alpha \neq 0, -1, -2, \ldots$.

Later Agarwal (1967) defined the operators

$$I_q^{\eta,\alpha} f(x) = \frac{x^{-\eta-\alpha}}{(1-q)^{1-\alpha}} \Pi_q \begin{bmatrix} \alpha; \\ 1 \end{bmatrix} \int_0^{x} [x - tq]_{\alpha-1}\, t^{\eta} f(t)\, d(t; q) \quad \ldots(1.2)$$

The extensions of AlSalam's (1966) and Agarwal's (1967) operators were given by Upadhyay (1970). The operators defined by her are as follows:

$$I_q [(a); (b); z, \eta : f(x)] = \frac{x^{-\eta-1}}{(1-q)} \int_0^x t^{\eta}\, {}_A\phi_B \begin{bmatrix} (a); & \frac{zt}{x} \\ (b); & \end{bmatrix} f(t)\, d(t; q) \quad \ldots(1.3)$$

and

$$K_q [(a); (b); z, \eta : f(x)] = \frac{x^{\eta}q^{-\eta}}{(1-q)} \int_x^{\infty} t^{-\eta-1}\, {}_A\phi_B \begin{bmatrix} (a); & \frac{zt}{x} \\ (b); & \end{bmatrix} f(t)\, d(t; q). \quad \ldots(1.4)$$

In this paper we introduce two more operators which may be regarded as extensions of Upadhyay's operators.

2. The I_q and K_q Operators

Here we introduce the following two generalized fractional q-integral operators:

$$I_q\left[\begin{matrix}(a):(d);(f);z_1,z_2,\eta_1,\eta_2:f(x,y)\\(b):(e);(g);\end{matrix}\right]$$

$$=\frac{x^{-\eta_1-1}y^{-\eta_2-1}}{(1-q)^2}\int_0^x\int_0^y t_1^{\eta_1}t_2^{\eta_2}\,\Phi\left[\begin{matrix}(a):(d);(f);\frac{z_1t_1}{x},\frac{z_2t_2}{y}\\(b):(e);(g);\end{matrix}\right]f(t_1,t_2)\,d(t_1;q)\,d(t;q)$$

$$=\sum_{k,j=0}^{\infty}q^{k(\eta_1+1)+j(\eta_2+1)}\,\Phi\left[\begin{matrix}(a):(d);(f);z_1q^k,z_2q^j\\(b):(e);(g);\end{matrix}\right]f(xq^k,yq^j) \qquad \ldots(2.1)$$

and

$$K_q\left[\begin{matrix}(a):(d);(f);z_1,z_2,\eta_1,\eta_2:f(x,y)\\(b):(e);(g);\end{matrix}\right]$$

$$=\frac{x^{\eta_1}y^{\eta_2}q^{-\eta_1-\eta_2}}{(1-q)^2}\int_x^\infty\int_y^\infty t_1^{-\eta_1-1}t_2^{-\eta_2-1}\,\Phi\left[\begin{matrix}(a):(d);(f);\frac{z_1x}{t_1},\frac{z_2y}{t_2}\\(b):(e);(g);\end{matrix}\right]$$

$$\times f(t_1,t_2)\,d(t_1;q)\,d(t_2;q)$$

$$=\sum_{k,j=0}^{\infty}q^{k\eta_1+j\eta_2}\,\Phi\left[\begin{matrix}(a):(d);(f);z_1q^{k+1},z_2q^{j+1}\\(b):(e);(g);\end{matrix}\right]f(xq^{-k-1},yq^{-j-1}). \qquad \ldots(2.2)$$

The operators I_q and K_q defined by Upadhyay (1970) and also by AlSalam (1966) and Agarwal (1967) are limiting cases of (2.1) and (2.2).

We now derive certain interconnecting theorems between these and the other operators of q-integration.

3. Relation of I_q, K_q with M_q-Operators

Theorem I — If $\sum_{\lambda_1,\lambda_2=-\infty}^{\infty}|q^{\lambda_1 s_1+\lambda_2 s_2}f(q^{\lambda_1},q^{\lambda_2})|$ converges, $|q|<1$, $|z_1|<1$, $|z_2|<1$ and $\mathrm{Re}(\eta_1-s_1)>-1$, $\mathrm{Re}(\eta_2-s_2)>-1$, then

$$M_q \left\{ I_q \begin{bmatrix} (a) : (d); (f); z_1, z_2, \eta_1, \eta_2 : f(x, y) \\ (b) : (e); (g); \end{bmatrix} \right\} = (1 - q^{\eta_1+1-s_1})^{-1} (1 - q^{\eta_2+1-s_2})^{-1}$$

$$\times \Phi \begin{bmatrix} (a) : (d), \eta_1 + 1 - s_1; (f), \eta_2 + 1 - s_2; z_1, z_2 \\ (b) : (e), \eta_1 + 2 - s_1; (g), \eta_2 + 2 - s_2; \end{bmatrix} M_q [f(x, y)] \qquad ...(3.1)$$

where a basic analogue of the Mellin transform of $f(x, y)$ is defined as

$$M_q [f(x, y)] = \int_0^\infty \int_0^\infty x^{s_1-1} y^{s_2-1} f(x, y)\, d(x; q)\, d(y; q) \qquad ...(3.2)$$

and the following theorem.

Theorem II — If $\sum\limits_{\lambda_1, \lambda_2 = -\infty}^{\infty} | q^{\lambda_1 s_1 + \lambda_2 s_2} f(q^{\lambda_1}, q^{\lambda_2}) |$ converges, $|q| < 1$, $|z_1| < 1$, $|z_2| < 1$ and $\text{Re}(\eta_1 + s_1) > 0$, $\text{Re}(\eta_2 + s_2) > 0$, then

$$M_q \left\{ K_q \begin{bmatrix} (a) : (d); (f); z_1, z_2, \eta_1, \eta_2 : f(x, y) \\ (b) : (e); (g); \end{bmatrix} \right\} = q^{s_1+s_2} (1 + q^{\eta_1+s_1})^{-1} (1 - q^{\eta_2+s_2})^{-1}$$

$$\times \Phi \begin{bmatrix} (a) : (d), \eta_1 + s_1; (f), \eta_2 + s_2; q, z_2 q \\ (b) : (e), \eta_1 + s_1 + 1; (g), \eta_2 + s_2 + 1 \end{bmatrix} M_q [f(x, y)] \qquad ...(3.3)$$

where $M_q [f(x, y)]$ is defined by (3.2).

Using the basic integrals given by Hahn and interchanging the summations and integrations under valid conditions, one can get the proof of (3.1) and (3.3).

Some other theorems are as follows (The proofs, being straightforward, are omitted):

Theorem III — If $\Phi [x_1, y_1, x_2, y_2]$

$$= I_q \begin{bmatrix} (a) : (d); (f); z_1, z_2, \eta_1, \eta_2 : [1 - x_1 x_2 q^{\alpha_1}]_{-\alpha_1} [1 - y_1 y_2 q^{\alpha_2}]_{-\alpha_2} f(x, y) \\ (b) : (e); (g); \end{bmatrix}$$

and

$$\psi(x_1, y_1) = I_q \begin{bmatrix} (a) : (d); (f); z_1, z_2, \eta_1, \eta_2 : h(x_1, y_1) \\ (b) : (e); (g); \end{bmatrix}$$

with

$$h(x_1, y_1) = \Pi_q \begin{bmatrix} x_1 q^{c_1}, & q^{1-c_1}/x_1, & y_1 q^{c_2}, & q^{1-c_2}/y_1; \\ x_1, & q/x_1, & y_1, & q/y_1; \end{bmatrix} f(x_1, y_1)$$

then

$$\psi(x_1, y_1) = (1 - q)^{-2} \Pi_q \begin{bmatrix} c_1, \alpha_1 - c_1, c_2, \alpha_2 - c_2; \\ \alpha_1, \quad 1, \alpha_2, \quad 1; \end{bmatrix}$$

$$\times \int_0^\infty \int_0^\infty x_2^{c_1-1} y_2^{c_2-1} \Phi [x_1, x_2, y_1, y_2]\, d(x_2; q)\, d(y_2; q) \qquad ...(3.4)$$

provided (i) $\text{Re}(\alpha_1) > \text{Re}(c_1) > 0$; $\text{Re}(\alpha_2) > \text{Re}(c_2) > 0$, $|z_1| < 1$, $|z_2| < 1$, $|q| < 1$, and (ii) the series representing the basic integral for $\Phi[x_1, y_1, x_2, y_2]$ and $\psi[x_1, x_2]$ converges absolutely.

Theorem IV — If $\Phi[x_1, x_2, y_1, y_2]$

$$= K_q\left[\begin{matrix}(a):(d);(f);z_1,z_2,\eta_1,\eta_2;[1-x_2q^{\alpha_1}/x_1]_{-\alpha_1}[1-y_2q^{\alpha_2}/y_1]_{-\alpha_2}f(x_1,y_1)\\(b):(e);\ (g);\end{matrix}\right]$$

and

$$\psi(x_1, y_1) = K_q\left[\begin{matrix}(a):(d);(f);z_1,z_2,\eta_1,\eta_2:h(x_1,y_1)\\(b):(e);\ (g);\end{matrix}\right]$$

with

$$h(x_1, y_1) = \Pi_q\left[\begin{matrix}q^{c_1}/x_1, & x_1q^{1-c_1}, & q^{c_2}/y_1, & y_1q^{1-c_2};\\ qx_1, & 1/x_1, & qy_1, & 1/y_1\end{matrix}\right]f(x_1, y_1)$$

then

$$\psi(x_1, y_2) = (1-q)^2\,\Pi_q\left[\begin{matrix}c_1, & \alpha_1 - c_1, & c_2, & \alpha_2 - c_2;\\ \alpha_1, & 1, & \alpha_2, & 1\end{matrix}\right]$$

$$\times\int_0^\infty\int_0^\infty x_2^{c_1-1}\,y_2^{c_2-1}\,\phi[x_1, y_1, x_2, y_2]\,d(x_2;q)\,d(y_2;q) \qquad \ldots(3.5)$$

provided (i) $\text{Re}(\alpha_1) > \text{Re}(c_1) > 0$; $\text{Re}(\alpha_2) > \text{Re}(c_2) > 0$, $|z_1| < 1$, $|z_2| < 1$, $|q| < 1$ and (ii) the series representing basic integrals for $\Phi[x_1, y_1, x_2, y_2]$ and $\psi(x_1, x_2)$ converge absolutely.

4. An Interconnection Between K_q and I_q Operators

Theorem V — If

$$\sum_{\lambda_1,\lambda_2=-\infty}^{\infty}|q^{\lambda_1(\eta_1+1)+\lambda_2(\eta_2+1)}f(q^{\lambda_1}, q^{\lambda_2})|$$

and

$$\sum_{\lambda_1,\lambda_2=-\infty}^{\infty}|q^{-\lambda_1\eta_1-\lambda_2\eta_2}g(q^{\lambda_1}, q^{\lambda_2})|$$

converge, $|q| < 1$, $|z_1| < 1$, $|z_2| < 1$, $\text{Re}(\eta_1) > -1$, $\text{Re}(\eta_2) > -1$, then

$$\int_0^\infty\int_0^\infty f(x,y)\,K_q\left[\begin{matrix}(a):(d);(f);z_1,z_2,\eta_1,\eta_2:g(x,y)\\(b):(e);\ (g);\end{matrix}\right]d(x;q)\,d(y;q) =$$

(*equation continued on p. 585*)

$$= \int_0^\infty \int_0^\infty g(xq^{-1}, yq^{-1})\, I_q \begin{bmatrix} (a):(d);(f);z_1q,\, z_2q,\, \eta_1,\eta_2 : f(x,y) \\ (b):\ (e);\ (g); \end{bmatrix} d(x;q)\, d(y;q). \quad \ldots(4.1)$$

Theorem VI — If Re $(\eta_1) > -1$, Re $(\eta_2) > -1$, $|q| < 1$, $|z_1| < 1$, $|z_2| < 1$, then

$$I_q \begin{bmatrix} (a):(d);(f);z_1, z_2, \eta_1, \eta_2 \\ (b):(e);\ (g); \end{bmatrix} : K_q \begin{bmatrix} (l)\ :(m);(n);z_1, z_2, \eta_1, \eta_2 : f(x,y) \\ (p):\ (u);(v); \end{bmatrix}\Bigg]$$

$$= K_q \begin{bmatrix} (l)\ :(m);(n);z_1, z_2, \eta_1, \eta_2, \\ (p):\ (u);(v); \end{bmatrix} : I_q \begin{bmatrix} (a):(d);(f);z_1, z_2, \eta_1, \eta_2 : f(x,y) \\ (b):(e);\ (g); \end{bmatrix}\Bigg].$$

5. I_q, K_q-Operators and the L_q-Operator

Theorem VII — If

$$\sum_{\lambda_1, \lambda_2 = -\infty}^{\infty} | q^{\lambda_1(\beta_1+1)+\lambda_2(\beta_2+1)} f(q^{\lambda_1}, q^{\lambda_2}) |$$

converges, $|q| < 1$, $|z_1| < 1$, $|z_2| < 1$, Re $(\eta_1 + \beta_1) > -1$, Re $(\eta_2 + \beta_2) > -1$, then,

$$\int_0^\infty \int_0^\infty f(x_2, y_2)\, I_q \begin{bmatrix} (a):(d);(f);z_1, z_2, \eta_1, \eta_2 : (x_1x_2)^{\beta_1} (y_1y_2)^{\beta_2}\, e_q(-x_1x_2)\, e_q(-y_1y_2) \\ (b):(e);\ (g); \end{bmatrix}$$

$$\times\, d(x_2;q)\, d(y_2;q)$$

$$= I_q \begin{bmatrix} (a):(d);(f);z_1, z_2, \eta_1, \eta_2 : x_1^{\beta_1} y^{\beta_2}\, L_{q,x_1,y_1}\{x_2^{\beta_1} y_2^{\beta_2} f(x_2, y_2)\} \\ (b):(e);\ (g); \end{bmatrix} \quad \ldots(5.1)$$

where

$$L_{q,x_1,y_1}(F(x_1, y_2)) = \int_0^\infty \int_0^\infty e_q(-x_1x_2)\, e_q(-y_1y_2)\, F(x_2, y_2)\, d(x_2;q)\, d(y_2;q).$$

Theorem VIII — If

$$\sum_{\lambda_1, \lambda_2 = -\infty}^{\infty} | q^{\lambda_1(\beta_1+1)+\lambda_2(\beta_2+1)} f(q^{\lambda_1}, q^{\lambda_2}) |$$

converges, $|z_1| < 1$, $|z_2| < 1$, $|q| < 1$, Re$(\eta_1 + \beta_1) > -1$, Re $(\eta_2 + \beta_2) > -1$, then

$$\int_0^\infty \int_0^\infty f(x_2, y_2)\, K_q \begin{bmatrix} (a) : (d); (f); z_1, z_2, \eta_1, \eta_2 : (x_1 x_2)^{\beta_1} (y_1 y_2)^{\beta_2} e_q(-x_1 x_2)\, e_q(-y_1 y_2) \\ (b) : (e); (g); \end{bmatrix}$$

$$\times\, d(x_2; q)\, d(y_2; q)$$

$$= K_q \begin{bmatrix} (a) : (d); (f); z_1, z_2, \eta_1, \eta_2 : x_1^{\beta_1} y_1^{\beta_2} L_{q, x_1, y_1} \{x_2^{\beta_1} y_2^{\beta_2} f(x, y)\} \\ (b) : (e); (g); \end{bmatrix}. \qquad \ldots(5.2)$$

6. Applications

In this section we give certain applications of the theorems and the operators given in the previous sections.

(i) Consider

$$I_q \begin{bmatrix} (a) : (b); (b'); q^{1+c-a-b}, q^{1+c'-a-b'}, \eta_1, \eta_2 : x^{\lambda-1} y^{\beta-1} \\ - : (c); (c^1); \end{bmatrix}$$

$$= x^{\lambda-1} y^{\beta-1} \sum_{k, j=0}^{\infty} q^{k(\eta_1+\lambda)+j(\eta_2+\lambda)}$$

$$\times\, \Phi \begin{bmatrix} a : b, b'; q^{1+c-a-b+k}, q^{1+c'-a-b'+j} \\ - : c, c'; \end{bmatrix}$$

$$= x^{\lambda-1} y^{\beta-1} (1 - q^{\eta_1+\lambda})^{-1} (1 - q^{\eta_2+\lambda})^{-1}$$

$$\times\, \Phi \begin{bmatrix} a : b, \quad \eta_1 + \lambda \quad ; b', \quad \eta_2 + \beta \quad ; q^{1+c-a-b}, q^{1+c'-a-b'} \\ - : c, \eta_1 + \lambda + 1 ; c', \eta_2 + \beta + 1 ; \end{bmatrix}. \qquad \ldots(6.1)$$

On the other hand

$$I_q \begin{bmatrix} a : b, b'; q^{1+c-a-b}, q^{1+c'-a-b'}, \eta_1, \eta_2 : x^{\lambda-1} y^{\beta-1} \\ - : c, c'; \end{bmatrix}$$

$$= x^{\lambda-1} y^{\beta-1} \sum_{k, j=0}^{\infty} q^{k(\eta_1+\lambda)+j(\eta_2+\beta)}$$

$$\times \sum_{m, n=0}^{\infty} \frac{(a; q)_m (aq^m; q)_n (b; q)_m (b'; q)_n}{(c; q)_m (c'; q)_n (q; q)_m (q; q)_n}\, q^{(1+c-a-b+k)m+(1+c'-a-b'+j)n}$$

$$= x^{\lambda-1} y^{\beta-1} (1 - q^{\eta_1+\lambda})^{-1} (1 - q^{\eta_2+\beta})^{-1}\, \Pi_q \begin{bmatrix} 1, 1 + c' - a - b' + \eta_2 + \beta; \\ 1 + c' - a - b', \eta_2 + \beta \end{bmatrix}$$

(equation continued on p. 587)

$$\times \sum_{m=0}^{\infty} \frac{(a;q)_m (b;q)_m (\eta_1+\lambda;q)_m (a+b'-c'-\eta_2-\beta;q)_m}{(c;q)_m (q;q)_m (\eta_1+\lambda+1;q)_m (a+b'-c';q)_m} q^{m(1+c-a-b-\eta_2-\beta)}$$

$$\times \sum_{r=0}^{\infty} \frac{(aq^m:q)_r bq^m;q)_r (\lambda+\eta_1+m;q)_r (a+1+c'+m;q)_r}{(cq^m;q)_r (q^{1+m};q)_r (\eta_1+\lambda+1+m;q)_r (c';q)_r (q;q)_r}$$

$$\times \frac{(\eta_2+\beta;q)_r (c'-b';q)_r}{(a+b'-c'+m;q)_r} q^{r(3-b'+c-a-b)}. \qquad \ldots(6.2)$$

Comparing (6.1) and (6.2) we get the following transformation:

$$\sum_{m=0}^{\infty} \frac{(a;q)_m (b;q)_m (\eta_1+\lambda;q)_m (a+b'-c'-\eta_2-\beta;q)_m}{(c;q)_m (q;q)_m (\eta_1+\lambda+1;q)_m (a+b'-c';q)_m} q^{m(1+c-a-b-\eta_2-\beta)}$$

$$\times {}_6\phi_5 \left[\begin{matrix} \eta_2+\beta, c'-b', a+m, b+m, \eta_1+\lambda+m, a+1+c'+m; q^{3-b'+c-a-b} \\ c' \quad , c+m, 1+m, \eta_1+\lambda+m+1, \ a+b'-c'+m; \end{matrix} \right]$$

$$= \Pi_q \left[\begin{matrix} 1+c'-a-b', \eta_2+\beta+1; \\ 1, 1+c'-a-b'+\eta_2+\beta \end{matrix} \right]$$

$$\times \Phi \left[\begin{matrix} a : b, \quad \eta_1+\lambda; b', \quad \eta_2+\beta; q^{1+c-a-b}, q^{1+c'-a-b'} \\ - : c, \eta_1+\lambda+1; c', \eta_2+\beta+1; \end{matrix} \right].$$

(ii) Also, one can easily equate $I_q\,[a:b,b';c,c';z_1,z_2,\eta_1,\eta_2:x^\lambda y^\mu]$ to two simultaneous values to get two equations by using Andrews' (1972) result:

$$\Phi\,[a:b,b';c,c';z_1q^k,z_2q^j]$$

$$= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m,n=\infty}^{\infty} \frac{(a;q)_{m+n}(b';q)_n (cq^m;q)_\infty}{(q;q)_m (q;q)_n (c';q)_n (bq^m;q)_\infty} (z_1q^k)^m (z_2q^j)^n.$$

Equating those two equations we get the following transformation:

$$\Phi \left[\begin{matrix} a : b, \eta_1+\lambda+1; b', \eta_2+\mu+1; z_1, z_2 \\ - : c, \eta_1+\lambda+2; c', \eta_2+\mu+2; \end{matrix} \right]$$

$$= \frac{(b;q)_\infty (az_1;q)_\infty}{(c;q)_\infty (z_1;q)_\infty} (1-q^{\eta_1+\lambda+1}) \sum_{n,r=0}^{\infty} \frac{(a;q)_n (b';q)_n}{(q;q)_n (q;q)_r}$$

$$\times \frac{(q^{\eta_2+\mu+1};q)_n (c-b;q)_r (z_1;q)_r z_2^n b^r}{(c';q)_n (q^{\eta_2+\mu+2};q)_n (az_1;q)_{n+r}}$$

$$\times {}_2\phi_1 \left[\begin{matrix} z_1q^r; q; q^{\eta_1+\lambda+1} \\ az_1q^{n+r} \end{matrix} \right].$$

(iii) Next, in (3.1), let

$$a = 1 - d_r + c_{r+1},\ z_1 = q^{d_r - c_{r+1}},\ z_2 = q^{d'_r - c'_{r+1}}$$

$$\eta_1 = c_{r+1} - 1,\ \eta_2 = c'_{r+1} - 1,\ f(x) = \Phi\begin{bmatrix} (c_r);\ \lambda x,\ \beta y \\ (d_{r-1}); \end{bmatrix}.$$

Then substituting these values in Theorem I we get the following transformation

$$\sum_{m,n=0}^{\infty} \frac{[1 - d_r + c_{r+1}; q]_{m+n}\, [q^{c_{r+1}}; q]_m\, [q^{c'_{r+1}}; q]_n}{[q; q]_m\, [q; q]_n\, [q^{c_{r+1}+1}; q]_m\, [q^{c'_{r+1}+1}; q]_m} M_q\, [\phi(\lambda x, \beta y)]$$

$$= (1 - q^{c_{r+1}})(1 - q^{c'_{r+1}})\, (1 - q^{c_{r+1} - \rho_1})^{-1}\, (1 - q^{c'_{r+1} - \rho_2})^{-1}$$

$$\times\ \Phi\begin{bmatrix} 1 - d_r + c_{r+1}:\ q^{c_{r+1} - \rho_1};\ q^{c'_{r+1} - \rho_2};\ q^{d_r - c_{r+1}},\ q^{d'_r - c'_{r+1}} \\ -:\ q^{c_{r+1} - \rho_1 + 1};\ q^{c_{r+1} - \rho_2 + 1} \end{bmatrix}$$

$$\times\ M_q\, [\phi'(\lambda x, \beta y)]$$

where

$$\phi(\lambda x, \beta y) = \phi\begin{bmatrix} (c_r)\ :\ q^{c_{r+1}+m};\ q^{c'_{r+1}+n};\ \lambda x,\ \beta y \\ (d_{r-1}):\ q^{c_{r+1}+m+1};\ q^{c'_{r+1}+n+1} \end{bmatrix},$$

and

$$\phi'(\lambda x, \beta y) = \phi\begin{bmatrix} (c_r);\ \lambda x,\ \beta y \\ (d_{r-1}) \end{bmatrix}$$

$M_q f(x)$ is the q-Mellin's transform of a function $f(x)$.

(iv) Lastly, in Theorem V, i.e. (4.1), take $a_1 = 1 - a$, $z_1 = q^{a-1}$, $z_2 = q^{b-1}$, remaining parameter equals to zero and

$$f(x, y) = x^{e_1 - 1}\, y^{e_2 - 1}\, \phi\, [d - e_1 - \eta_1 : xq^{e_1},\ xq^{e_2}]$$

$g(x, y) = \phi\, [a + \eta_1 : q^{-d_1}/x,\ q^{-d_2}/y]$, then we get the following relation between two q-double integrals

$$\sum_{m,n=0}^{\infty} \frac{[1 - a; q]_{m+n}\, [q^{\eta_1}; q]_m\, [q^{\eta_2}; q]_n}{[q; q]_m\, [q; q]_n\, [q^{\eta_1+1}; q]_m\, [q^{\eta_2+1}; q]_n}\, q^{am+bn}$$

$$\times \int_0^{\infty}\int_0^{\infty} \Phi\begin{bmatrix} q^{d_1 - e_1 - \eta_1};\ xq^{e_1},\ yq^{e_2} \\ -; \end{bmatrix}$$

$$\times\ \Phi\begin{bmatrix} a + \eta_1 :\ \eta_1 + m,\ \eta_2 + n;\ q^{1-d_1}/x,\ q^{1-d_2}/y \\ -:\ \eta_1 + m + 1,\ \eta_2 + n + 1 \end{bmatrix} d(x; q)\, d(y; q) =$$

(equation continued on p. 589)

$$= \frac{(1-q^{\eta_1})(1-q^{\eta_2})}{(1-q^{\eta_1+e_1})(1-q^{\eta_2+e_2})}$$

$$\times \sum_{m,n=0}^{\infty} \frac{(1-q;q)_{m+n}(q^a;q)_m(q^b;q)_n}{(q;q)_m(q;q)_n(q^{\eta_1+e_1+1};q)_m}$$

$$\times \frac{(q^{\eta_1+e_1};q)_m(q^{\eta_2+e_2};q)_n}{(q^{\eta_2+e_2+1};q)_n} q^{(a-1)m+(b-1)n}$$

$$\times \int_0^\infty \int_0^\infty \Phi\begin{bmatrix} a+\eta_1: q^{-d_1}/x, q^{-d_2}/y \\ - \quad : \end{bmatrix}$$

$$\times \Phi\begin{bmatrix} 1-e_1-\eta_1 : q^{\eta_1+e_1+m}; q^{\eta_2+e_2+n}; xq^{e_1}, yq^{e_2} \\ - \quad : q^{\eta_1+e_1+m+1}; q^{\eta_2+e_2+n+1} \end{bmatrix} d(x;q)\, d(y;q).$$

The above result holds, provided Re (η_1), Re $(\eta_2) > 0$.

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REFERENCES

Agarwal, R. P. (1967). Certain fractional q-integrals and q-derivatives. *Proc. Camb. phil. Soc.*, **63**, 727-34.

AlSalam, W. A. (1966). Some fractional q-integrals and q-derivatives. *Proc. Edinb. Math. Soc.*, **15**, 135-40.

Andrews, G. E. (1972). Summation and transformation for basic Appell series. *J. Lond. math. Soc.*, **4**, 618-22.

Upadhyay, M. (1970). Certain fractional q-integral operators. Thesis approved for Ph.D. degree of Lucknow University, Lucknow.