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ON THE APPLICATION OF RADIATION CONDITION FOR A SOURCE IN A ROTATING STRATIFIED FLUID

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In this paper it is seen that the stratification effect on the inertia terms represented by the term $\beta(\partial P/\partial z)$ in equation derived by Sarma and Naidu (1972) is responsible for the poles n_2 and n_2' to lie in the upper half n-plane. If $\beta(\partial P/\partial z)$ were neglected (the case of Bounsinessq fluid), both the poles would lie on the path of integration in which case Lighthill's radiation condition (which requires the path of integration to pass below the negative pole and above the positive pole for $z > 0$) should be used. Physically this condition corresponds to the absence of any source at infinity generating inward travelling waves. Such waves have zero amplitude for finite r, in the present problem because of the presence of dissipative term $\beta(\partial P/\partial z)$ and hence there is no need to apply radiation condition.

In this paper, the author asserts that the stratification effect on the inertia terms represented by $\beta(\partial P/\partial z)$ in eqn. (1) of section 1, is responsible for the poles n_2, n_2' to lie in the upper half n-plane while performing Fourier inverse transform. If $\beta(\partial P/\partial z)$ were neglected both the poles would lie on the path of integration in which case Lighthills's radiation condition (which requires the path of integration to pass below the negative pole and above the positive pole for $z > 0$) should be used. Physically this condition corresponds to the absence of any source at infinity, generating inward travelling waves. Such waves have zero amplitude for finite r, in the present problem because of the presence of dissipative term $\beta(\partial P/\partial z)$ in the governing eqn. (1) and, hence there is no need to apply radiation condition. In fact the solution obtained by Sarma and Naidu satisfies Sommerfeld's radiation condition namely

$$\lim_{r\to\infty} r\left(\frac{\partial P}{\partial r} + KP\right) = 0.$$

1. GOVERNING EQUATION

The governing equation for the perturbation pressure P for the flow generated by an oscillatory point source in an unbounded rotating stratified fluid, when the

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stratification effect on the inertia terms is taken into account is (Sarma and Naidu 1972)

$$\frac{\partial}{\partial t}\left[\left(\frac{\partial^2}{\partial t^2}+N^2\right)\left(\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}\right)+\left(\frac{\partial^2}{\partial t^2}+4\Omega^2\right)\left(\frac{\partial^2 P}{\partial Z^2}+\beta\frac{\partial P}{\partial Z}\right)\right]$$
$$=\rho_0\left(\frac{\partial^2}{\partial t^2}+N^2\right)\left(\frac{\partial^2}{\partial t^2}+4\Omega^2\right)qe^{i\omega t}\delta(x)\,\delta(y)\,\delta(z) \qquad \ldots(1)$$

where Ω is the angular velocity of rotation of the fluid, $qe^{i\omega t}$ is the strength of the source (ω being the frequency of oscillation of the source), $\delta(x)$ is the Dirac delta function and ρ_0 is the basic density distribution given by

$$\rho_0=\rho_0'\exp(-\beta z) \qquad \ldots(2)$$

(ρ_0' and β being the characteristic density and stratification parameter respectively) so that the Brunt-Vaisala frequency

$$N=\left(-\frac{g}{\rho_0}\frac{d\rho_0}{dz}\right)^{1/2}=(\beta g)^{1/2} \qquad \ldots(3)$$

is constant. The density profile (2) cannot be assumed to exist throughout the unbounded fluid, since $\rho_0\to 0$ as $z\to\infty$ and $\rho_0\to\infty$ as $z\to-\infty$. So we assume that the density profile (2) is valid for all finite z (i.e., for all $|z|\leqslant Z$ where Z is not very large); and as $z\to\infty$ (or $-\infty$), ρ_0 approaches slowly, smoothly and monotonically some positive finite value, say ρ_0^+ (or ρ_0^-).

For the region $z>0$ (i.e., for $0\leqslant\theta<\pi/2$ where r,θ,ϕ are spherical polars), solutions of eqn. (1) have been obtained in closed form for the elliptic ($2\Omega \lessgtr \omega \gtrless N$) and the hyperbolic ($2\Omega \lessgtr \omega \lessgtr N$) cases by Sarma and Naidu (1972). In this paper we are concerned with the hyperbolic case only.

2. Solution

The solution of (1), following Lighthill (1960), is given in the form of three-fold Fourier integral:

$$P=\frac{-i\rho_0' q(N^2-\omega^2)\exp(i\omega t)}{8\pi^3\omega}\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp(i(lx+my+nz))\,dl\,dm\,dn}{G(l,m,n)} \qquad \ldots(4)$$

where

$$G=\left(n^2-i\beta n-\frac{s^2}{\lambda^2}\right),\ s^2=l^2+m^2,\ \lambda^2=\frac{(4\Omega^2-\omega^2)}{(\omega^2-N^2)}(>0). \qquad \ldots(5)$$

The integral in (4) is evaluated by performing n-integration by the method of residues and l and m integrations by changing the variables x, y, z to spherical polars (r, θ, ϕ). The zeros of G are

$$n_2, n_2' = (\tfrac{1}{2}) \left(i\beta \pm \left(\frac{4s^2}{\lambda^2} - \beta^2 \right)^{1/2} \right). \qquad \ldots(6)$$

The integrand of

$$\int_{-\infty}^{\infty} \frac{\exp(inz)\, dn}{G} \qquad \ldots(7)$$

has two poles lying in the upper half n-plane and none in the lower half plane (see Figs. 1 and 2). For the flow region $z > 0$, by considering a semicircular contour C in the upper half n-plane (imaginary part of $n > 0$) (shown in Figs. 1 and 2), it can be written as

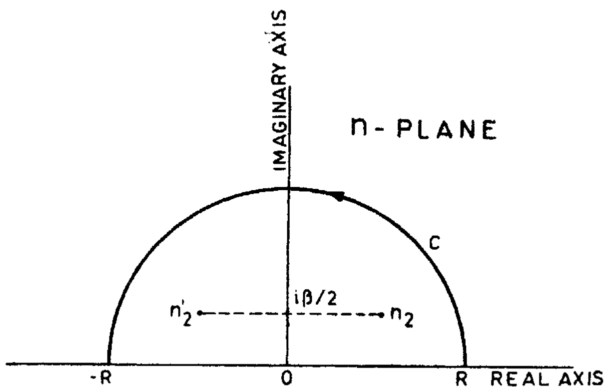

FIG. 1. When $s^2 > \beta^2 \lambda^2/4$, $n_2, n_2' = \frac{1}{2}[i\beta \pm ((4s^2/\lambda^2) - \beta^2)^{1/2}]$.

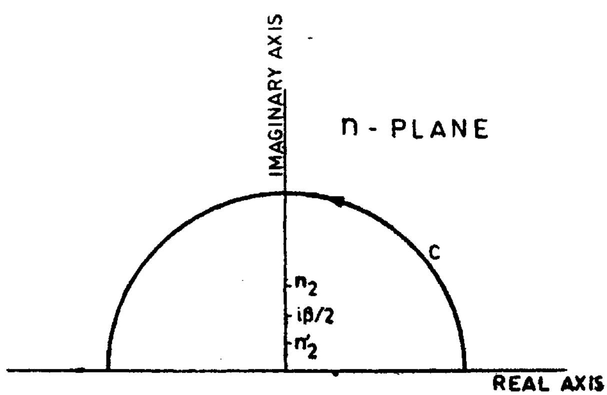

FIG. 2. When $0 < s^2 < \beta^2 \lambda^2/4$, $n_2, n_2' = i/2\, [\beta \pm (\beta^2 - (4s^2/\lambda^2))^{1/2}]$.

$$\int_{-R}^{R} \frac{\exp(inz)\,dn}{G} + \int_{C} \frac{\exp(inz)\,dn}{G}$$

$$= 2\pi i \text{ (sum of the residues at the poles } n_2 \text{ and } n_2'). \quad \text{...(8)}$$

As $R \to \infty$ the integral over C in (8) becomes zero. Hence integral (7) is evaluated from the contribution of the poles n_2 and n_2' and the triple integral (4) now becomes

$$P = \frac{iq\rho_0'(N^2 - \omega^2)\exp(i\omega t) - \beta z/2)}{4\pi^2\omega}$$

$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\sin\left(z\left(\frac{s^2}{\lambda^2} - \frac{\beta^2}{4}\right)^{1/2}\right)\exp i(lx + my))\,dl\,dm}{\left(\frac{s^2}{\lambda^2} - \frac{\beta^2}{4}\right)^{1/2}} \quad \text{...(9)}$$

and this has been evaluated (Sarma and Naidu 1972) and the final expression for the perturbation pressure in the region $z > 0$ is

$$P = 0 \quad \text{for} \quad \tan\theta > 1/\lambda \quad \text{...(10}a\text{)}$$

$$P = \frac{iq\rho_0'(\omega^2 - 4\Omega^2)\exp(i\omega t - (\tfrac{1}{2})\beta r\cos\theta)\cosh(\beta r\sqrt{M}/2)}{2\pi\omega r\sqrt{M}}$$

$$\text{for} \quad \tan\theta < 1/\lambda \quad \text{...(10}b\text{)}$$

where $M = \cos^2\theta - \lambda^2\sin^2\theta$.

For the region $z < 0$, taking a semicircular contour in the lower half n-plane (which encloses no poles) it is found that

$$P = 0. \quad \text{...(10}c\text{)}$$

It has been observed by Rao (1973) that the solution (10) is wrong since radiation condition has not been applied. In this context, the author asserts that the stratification effect on the inertia terms [represented by the term $\beta(\partial P/\partial z)$ in eqn. (1)] is responsible for the poles n_2 and n_2' to lie in the upper half n-plane (see Figs. 1 and 2). If $\beta(\partial P/\partial z)$ were neglected, both the poles would lie on the path of integration in which case Lighthill's radiation condition (which requires the path of integration to pass below the negative pole and above the positive pole for $z > 0$) should be used. Physically, this condition corresponds to the absence of any source at infinity generating inward travelling waves. Such waves have zero amplitude for finite r, in the present problem, because of the presence of the dissipative term $\beta(\partial P/\partial z)$; and hence there is no need to apply radiation condition (see Carrier *et al.* 1966).

For large r, inside the cone (i.e., $\tan\theta < 1/\lambda$),

$$P \simeq \frac{iq\rho_0'(\omega^2 - 4\Omega^2)\exp(i\omega t)\exp(-Kr)}{4\pi\omega r\sqrt{M}} \quad \text{...(11a)}$$

where

$$K = (\beta/r)(\cos\theta - \sqrt{M})\ (\geqslant 0) \quad \text{...(11b)}$$

which tends to zero as $r \to \infty$.

Since P is a non-wavy disturbance which vanishes at infinity, there is no need to verify whether P satisfies Sommerfeld's radiation condition. However, it can be easily seen that

$$\lim_{r\to\infty} r((\partial P/\partial r) + KP) = 0. \quad \text{...(12)}$$

Rao (1973) has applied Lighthill's radiation condition as follows:

By putting $n = (\frac{1}{2})\, i\beta + n'$ in (4) 'the poles of the integrand are made to lie on the real axes' and (4) becomes

$$P = \frac{-i\rho_0' q(N^2 - \omega^2)\exp(i\omega t - \beta z/2)}{8\pi^3\omega}$$

$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(i(lx + my))\, dl\, dm \int_{-\infty}^{\infty} \frac{\exp(in'z)}{(n'^2 - n_1^2)}\, dn \quad \text{...(13)}$$

where

$$n_1 = \left(\frac{s^2}{\lambda^2} - \frac{\beta^2}{4}\right)^{1/2}. \quad \text{...(14)}$$

Then by applying Lighthill's radiation condition (wherein for $z > 0$, only the pole $-n_1$ contributes to n'-integration) and by using the substitutions

$$l = s\cos\psi,\ m = s\sin\psi;\ x = R\cos\phi,\ y = R\sin\phi \quad \text{...(15)}$$

integral (13) becomes

$$P = \frac{-\rho_0' q(N^2 - \omega^2)\exp\left(i\omega t - \frac{\beta z}{2}\right)}{4\pi\omega}$$

$$\times \int_0^{\infty} \frac{\exp\left(-iz\left(\frac{s^2}{\lambda^2} - \frac{\beta^2}{4}\right)^{1/2}\right)}{\left(\frac{s^2}{\lambda^2} - \frac{\beta^2}{4}\right)^{1/2}} J_0(Rs)\, s\, ds. \quad \text{...(16)}$$

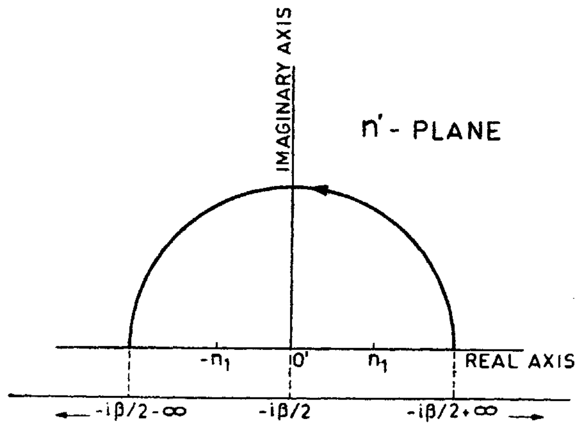

FIG. 3. When $s^2 > \beta^2\lambda^2/4$, $n_1 = ((s^2/\lambda^2) - (\beta^2/4))^{1/2}$.

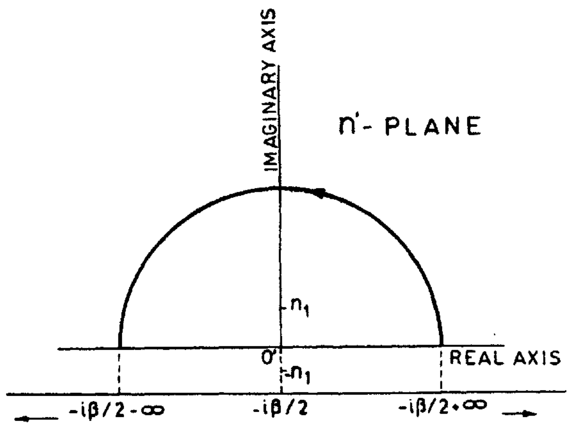

FIG. 4. When $0 < s^2 < \beta^2\lambda^2/4$, $n_1 = i((\beta^2/4) - (s^2/\lambda^2))^{1/2}$.

The limits for n'-integration are $(-(i\beta/2) - \infty, -(i\beta/2) + \infty)$ but not $(-\infty, \infty)$ [the latter being wrongly used by Rao in eqn. (13)]. Moreover n_1 is assumed (by Rao 1973) to be real which contradicts the integral in (16). For $0 < s^2 < (\beta^2/4)\lambda^2$, n_1 is purely imaginary and the poles n_1 and $-n_1$ lie in the upper and lower halves of the n'-plane (see Fig. 4); although, for $s^2 > (\beta^2/4)\lambda^2$, n_1 and $-n_1$ lie on the real axis of the n'-plane (see Fig. 3). Therefore replacing of the path of integration $(-(i\beta/2) - \infty, -(i\beta/2) + \infty)$ by the path $(-\infty, \infty)$, under the assumption that n_1, $-n_1$ lie on the real axis of n'-plane, is wrong since either the path $(-\infty, \infty)$ passes through the singularities n_1, $-n_1$ (for $s^2 > (\beta^2/4)\lambda^2$) or n_1, $-n_1$ lie on either side of the path $(-\infty, \infty)$ (for $0 < s^2 < (\beta^2/4)\lambda^2$).

Again, by an alternative procedure (using Sommerfeld's radiation condition) Rao (1973) has obtained the following integral for the disturbance pressure:

$$P = \frac{-Ki \exp(i\omega t)}{2\pi} \int_{-\infty}^{\infty} [J_0(\lambda\gamma r) - iY_0(\lambda\gamma r)] \exp(-inz)\, dn \qquad \ldots(17)$$

where $\gamma^2 = n^2 + i\beta n$.

Putting

$$\gamma^2 = n^2 + i\beta n = s^2 + (\tfrac{1}{4})\,\beta^2$$

in (17) he arrives at

$$P = \frac{-iK \exp\left(i\omega t - \frac{\beta z}{2}\right)}{\pi}$$

$$\times \int_0^{\infty} \left[J_0\left(\lambda r \left(s^2 + \frac{\beta^2}{4}\right)^{1/2}\right) - iY_0\left(\lambda r\left(s^2 + \frac{\beta^2}{4}\right)^{1/2}\right) \cos(zs)\right] ds. \qquad \ldots(18)$$

Here again the limits for s are wrongly taken to be

$$(-\infty, \infty) \text{ instead of } \left(\frac{i\beta}{2} - \infty,\ \frac{i\beta}{2} + \infty\right).$$

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