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ON MODIFIED DEFICIENCIES OF MEROMORPHIC FUNCTIONS

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In this paper the results of Toda (1970) have been extended to higher derivatives. The bounds for $\frac{T_\alpha(r, f^{(l)})}{T_\alpha(r, f)}$ have been obtained. Also the bounds for

$$K_\alpha(f') = \limsup_{r\to\infty} \frac{N_\alpha(r, f') + N_\alpha\left(r, \frac{1}{f'}\right)}{T_\alpha(r, f')}$$ in terms of modified defects have been given. It has been shown that, if $\sum_i \delta_\alpha(a_i) = \beta$ and $\delta_\alpha(\infty) = 2 - \beta$, then

$$\frac{\beta - 1}{\beta} \leqslant K_\alpha(f') \leqslant \frac{2(\beta - 1)}{\beta}.$$

INTRODUCTION

Let $f(z)$ be a meromorphic function in the finite plane $|z| < \infty$. If the order of $f(z)$ is infinite then the so-called exceptional set appears in the second fundamental theorem of Nevanlinna. Therefore in contrast to the case of finite order, we have to face many difficulties in the investigation of value distribution of meromorphic functions of infinite order. To avoid some of them, Toda (1970) introduced modified characteristic function and deficiencies to the Nevanlinna theory. He defines $T_\alpha(r, f)$, $N_\alpha(r, f)$, $\delta_\alpha(a, f)$, etc., as follows.

SECTION 1

Let $f(z)$ be a meromorphic function in $|z| < \infty$ of order ρ $(0 \leqslant \rho \leqslant \infty)$ and lower order μ. Denote by α any non-negative number smaller than ρ if ρ is not zero, and zero if $\rho = 0$. The symbols $T(r, f)$, $m(r, a)$, $N(r, a)$, $N(r, f)$ etc., have the usual meanings of the Nevanlinna theory of meromorphic functions (Hayman 1964). If r_0 is any positive number, then

$$T_\alpha(r, r_0, f) = \int_{r_0}^{r} \frac{T(t, f)}{t^{1+\alpha}}\, dt, \quad N_\alpha(r, r_0, f) = \int_{r_0}^{r} \frac{N(t, a)}{t^{1+\alpha}}\, dt$$

$$m_\alpha(r, r_0 f) = \int_{r_0}^{r} \frac{m(t, a)}{t^{1+\alpha}}\, dt, \quad \bar{N}_\alpha(r, r_0, f) = \int_{r_0}^{r} \frac{\bar{N}(t, a)}{t^{1+\alpha}}\, dt$$

and $$S_\alpha(r, r_0, f) = \int_{r_0}^{r} \frac{S(t, f)}{t^{1+\alpha}}\, dt.$$

For any complex number a, finite or not,

$\delta_\alpha(a, f)$, $\triangle_\alpha(a, f)$ and $\Theta_\alpha(a, f)$ are defined as follows :

$$\delta_\alpha(a, f) = \liminf_{r\to\infty} \frac{m_\alpha(r, a)}{T_\alpha(r, f)},\quad \triangle_\alpha(a, f) = \limsup_{r\to\infty} \frac{m_\alpha(r, a)}{T_\alpha(r, f)}$$

$$\Theta_\alpha(a, f) = 1 - \limsup_{r\to\infty} \frac{\bar{N}_\alpha(r, a)}{T_\alpha(r, f)}.$$

Toda (1970) proves the following :

Theorem A — Let $f(z)$ be a transcendental meromorphic fucntion in $|z| < \infty$, then

$$\sum_{a\neq\infty} \delta_\alpha(a) \leqslant \liminf_{r\to\infty} \frac{T_\alpha(r, f')}{T_\alpha(r, f)} \leqslant \limsup_{r\to\infty} \frac{T_\alpha(r, f')}{T_\alpha(r, f)} \leqslant 2 - \Theta_\alpha(\infty).$$

We extend the above result to the higher derivatives as follows :

Theorem 1 — If $f(z)$ is a transcendental meromorphic function in $|z| < \infty$, then for any positive integer

$$\sum_{a\neq\infty} \delta_\alpha(a) \leqslant \liminf_{r\to\infty} \frac{T_\alpha(r, f^{(l)})}{T_\alpha(r, f)} \leqslant \limsup_{r\to\infty} \frac{T_\alpha(r, f^{(l)})}{T_\alpha(r, f)} \leqslant (l+1) - l\,\Theta_\alpha(\infty).$$

To prove this we require the following lemma.

Lemma 1 — If $f(z)$ is a transcendental meromorphic function and $a_1, a_2, a_3, \ldots, a_q$ are distinct elements, then

$$\sum_{i=1}^{q} m_\alpha(r, a_i, f) + N_\alpha\left(r, \frac{1}{f^{(l)}}\right) \leqslant T_\alpha(r, f^{(l)}) + S_\alpha(r, f).$$

PROOF : Without loss of generality we may assume that $q \geqslant 2$.

Let $$F(z) = \sum_{i=1}^{q} \frac{1}{f(z) - a_i},$$

then $$m(r, F) \geqslant \sum_{i=1}^{q} m(r, a_i, f) - q \log \frac{3q}{\delta} - \log r.$$

Thus $$\sum_{i=1}^{q} m(r, a_i, f) \leqslant m(r, F) + O(1) \leqslant m\left(r, \frac{1}{f^{(l)}}\right) + S(r, f).$$

Hence, dividing by $t^{1+\alpha}$ and integrating from r_0 to r we get,

$$\sum_{i=1}^{q} m_\alpha(r, a_i f) \leqslant m_\alpha\left(r, \frac{1}{f^{(l)}}\right) + S_\alpha(r, f).$$

By adding $N_\alpha\left(r, \frac{1}{f^{(l)}}\right)$ to both the sides we get

$$\sum_{i=1}^{q} m_\alpha(r, a_i, f) + N_\alpha\left(r, \frac{1}{f^{(l)}}\right) \leqslant T_\alpha\left(r, \frac{1}{f^{(l)}}\right) + S_\alpha(r, f).$$

Hence $$\sum_{i=1}^{q} m_\alpha(r, a_i, f) + N_\alpha\left(r, \frac{1}{f^{(l)}}\right) \leqslant T_\alpha(r, f^{(l)}) + S_\alpha(r, f).$$

Proof of Theorem 1 : By Hayman (1964, p. 56) we know that

$$m(r, f^{(l)}/f) = S(r, f).$$

Hence $$m_\alpha(r, f^{(l)}/f) = S_\alpha(r, f).$$

Thus we have $m_\alpha(r, f^{(l)}) \leqslant m_\alpha(r, f) + S_\alpha(r, f)$.

Also $$N(r, f^{(l)}) = N(r, f) + l\bar{N}(r, f).$$

Hence $$N_\alpha(r, f^{(l)}) = N_\alpha(r, f) + l\bar{N}_\alpha(r, f).$$

So, $$T_\alpha(r, f^{(l)}) \leqslant T_\alpha(r, f) + l\bar{N}_\alpha(r, f) + S_\alpha(r, f).$$

Hence $$\frac{T_\alpha(r, f^{(l)})}{T_\alpha(r, f)} \leqslant 1 + l\,\frac{\bar{N}_\alpha(r, f)}{T_\alpha(r, f)} + O(1)$$

$$\therefore \quad \limsup_{r\to\infty} \frac{T_\alpha(r, f^{(l)})}{T_\alpha(r, f)} \leqslant (l + 1) - l\,\Theta_\alpha(\infty, f). \qquad \ldots(1)$$

On the other hand by Lemma 1 we have,

$$\sum_{i=1}^{q} m_\alpha(r, a_i, f) \leqslant T_\alpha(r, f^{(l)}) + S_\alpha(r, f).$$

Hence

$$\sum_{i=1}^{q} \delta_\alpha(a_i) \leqslant \liminf_{r\to\infty} \frac{T_\alpha(r, f^{(l)})}{T_\alpha(r, f)}. \qquad \ldots(2)$$

Now from (1) and (2) we have

$$\sum_{i=1}^{q} \delta_\alpha(a_i) \leqslant \liminf_{r\to\infty} \frac{T_\alpha(r, f^{(l)})}{T_\alpha(r, f)} \leqslant \limsup_{r\to\infty} \frac{T_\alpha(r, f^{(l)})}{T_\alpha(r, f)}$$
$$\leqslant (l+1) - l\,\text{Ⓗ}_\alpha(\infty)$$

i.e.
$$\sum_{a\neq\infty} \delta_\alpha(a) \leqslant \liminf_{r\to\infty} \frac{T_\alpha(r, f^{(l)})}{T_\alpha(r, f)} \leqslant \limsup_{r\to\infty} \frac{T_\alpha(r, f^{(l)})}{T_\alpha(r, f)}$$
$$\leqslant (l+1) - l\,\text{Ⓗ}_\alpha(\infty).$$

Remark 1 : The above inequality is sharp for $f(z) = \frac{e^z}{z}$.

Corollary 1.1 — If $f(z)$ is any meromorphic function with $\sum_{a\neq\infty} \delta_\alpha(a) = 1$ and $\delta_\alpha(\infty) = 1$, then

$$T_\alpha(r, f^{(l)}) \sim T_\alpha(r, f).$$

Corollary 1.2 — If $l = 1$ and $\Sigma\, \delta_\alpha(a) = 2$, then

$$\lim_{r\to\infty} \frac{T_\alpha(r, f^{(l)})}{T_\alpha(r, f)} = 2 - \text{Ⓗ}_\alpha(\infty, f).$$

Theorem 2 — If $f(z)$ is a transcendental meromorphic function in $|z| < \infty$, then

$$\frac{1}{(l+1) - l\,\text{Ⓗ}_\alpha(\infty)} \sum_{a\neq\infty} \delta_\alpha(a) \leqslant \delta_\alpha(0, f^{(l)}).$$

Proof follows by Lemma 1 and Theorem 1.

Remark 2 : The inequality in Theorem 2 is sharp for $f(z) = \frac{e^z}{z}$.

Corollary 2.1 — If $f(z)$ is a meromorphic function with $\delta_\alpha(\infty) = 1$, then $\sum_{a\neq\infty} \delta_\alpha(a) \leqslant \delta_\alpha(0, f^{(l)})$.

Corollary 2.2 — If $f(z)$ is a meromorphic function with at least one modified deficient value then o is the modified deficient value of $f^{(l)}$.

Singh and Sarangi (1973) have proved the following result.

Theorem B — If $f(z)$ is a meromorphic function of finite order ρ with $\sum_{a\in\bar{c}} \delta_\alpha(a, f^{(k)}) = 2$, then ρ is a positive integer.

Here we prove the following result even if the order of $f(z)$ is infinite.

Theorem 3 — If $f(z)$ is a meromorphic function in $|z| < \infty$ with $\sum_a \delta_\alpha(a, f') = 2$, then the order ρ of f is a positive integer if $\rho < \infty$ or ρ is infinite. To prove the theorem we require the following lemma.

Lemma 2 — If $f(z)$ is a meromorphic function in $|z| < \infty$ with $\sum_{a \neq \infty} \delta_\alpha(a) = 1$ and $\delta_\alpha(\infty) = 1$, then

$$\delta_\alpha(0, f') = 1 \text{ and } \delta_\alpha(\infty, f') = 1.$$

PROOF : By Toda (1970) we know that

$$\frac{1}{2 - \text{Ⓗ}_\alpha(\infty, f)} \sum_{a \neq \infty} \delta_\alpha(a) \leqslant \delta_\alpha(0, f').$$

Since $\quad 1 = \delta_\alpha(\infty) \leqslant \text{Ⓗ}_\alpha(\infty) \leqslant 1, \sum_{a \neq \infty} \delta_\alpha(a) \leqslant \delta_\alpha(0, f').$

Since $\sum_{a \neq \infty} \delta_\alpha(a) = 1$ we have $\delta_\alpha(0, f') = 1$.

By Theorem 1, we have

$$T_\alpha(r, f') \sim T_\alpha(r, f). \qquad \ldots(3)$$

Also we know that $N(r, f') \leqslant 2N(r, f)$

$$\therefore \quad N_\alpha(r, f') \leqslant 2N_\alpha(r, f).$$

Now from (3) and $\delta_\alpha(\infty) = 1$ it follows that

$$\delta_\alpha(\infty, f') = 1.$$

PROOF OF THEOREM 3 : Since

$$2 = \Sigma\, \delta_\alpha(a, f') \leqslant \Sigma\, \text{Ⓗ}_\alpha(a, f') \leqslant 2,$$

We have $\quad \delta_\alpha(\infty, f') = \text{Ⓗ}_\alpha(\infty, f')$

and $\quad N_\alpha(r, \infty, f') \geqslant 2\bar{N}_\alpha(r, \infty, f) = 2\bar{N}_\alpha(r, \infty, f')$

$$\therefore \quad 1 - \delta_\alpha(\infty, f') \geqslant 2(1 - \text{Ⓗ}_\alpha(\infty, f')).$$

Hence $\quad \delta(\infty, f') = 1.$

Now by Lemma 2 we have $\delta_\alpha(0, f'') = \delta_\alpha(\infty, f'') = 1$.

Hence, $\quad K_\alpha(f'') \leqslant 2 - \delta_\alpha(0, f'') - \delta_\alpha(\infty, f'') = 0,$

where $\quad K_\alpha(f'') = \limsup_{r \to \infty} \dfrac{N_\alpha(r, f'') + N_\alpha\left(r, \dfrac{1}{f''}\right)}{T_\alpha(r, f'')}.$

Hence by Toda (1970) ρ is a positive integer if $\rho < \infty$ or ρ is infinite.

Section 2

In this section we prove theorems on sum of the modified deficiencies.

Theorem 4 — Let $f(z)$ be a meromorphic function in $|z| < \infty$ with $\{a_i\}$ as modified deficient values such that $\Sigma\, \delta_\alpha(a_i) = \beta\, (a_i \neq \infty)$, $\Sigma\, \delta_\alpha(a_i) = 2$ where a_i's are distinct $(0 \leqslant |a_i| \leqslant \infty)$, then $T_\alpha(r, f') \sim T_\alpha(r, f)$.

PROOF : From the inequality of Milloux (1946), we have

$$pT(r, f) < T(r, f') + \sum_{1}^{p} N(r, a_i) - N\left(r, \frac{1}{f'}\right) + S(r, f).$$

Hence $$pT_\alpha(r, f) < T_\alpha(r, f') + \sum_{1}^{p} N_\alpha(r, a_i) - N_\alpha\left(r, \frac{1}{f'}\right) + S_\alpha(r, f).$$

Now dividing by $T_\alpha(r, f)$ and simplifying we get

$$\sum_{1}^{p} \delta_\alpha(a_i, f) \leqslant \Delta_\alpha(0, f') \liminf_{r\to\infty} \frac{T_\alpha(r, f')}{T_\alpha(r, f)}.$$

Now given $\epsilon > 0$ we can choose $a_1, a_2, \ldots, a_p$ $(p \geqslant 3)$.

So that $$\sum_{p+1}^{\infty} \delta_\alpha(a_i) < \epsilon \; (a_i \neq \infty).$$

Hence $$\sum_{1}^{p} \delta_\alpha(a_i) > \beta - \epsilon \; (a_i \neq \infty).$$

Since $1 \leqslant \beta \leqslant 2$ and $\delta_\alpha(\infty) = 2 - \beta$, it follows that

$$\sum_{1}^{p} \delta_\alpha(a_i) > \beta - \epsilon > 0.$$

So, $$\frac{\beta - \epsilon}{\Delta_\alpha(0, f')} \leqslant \liminf_{r\to\infty} \frac{T_\alpha(r, f')}{T_\alpha(r, f)}. \qquad \ldots(4)$$

Also we know that $\limsup_{r\to\infty} \frac{T_\alpha(r, f')}{T_\alpha(r, f)} \leqslant 2 - \Theta_\alpha(\infty, f) \leqslant \beta$.

Hence from (4) we have

$$\frac{\beta - \epsilon}{\Delta_\alpha(0, f')} \leqslant \liminf_{r\to\infty} \frac{T_\alpha(r, f')}{T_\alpha(r, f)} \leqslant \limsup_{r\to\infty} \frac{T_\alpha(r, f')}{T_\alpha(r, f)} \leqslant \beta.$$

Thus $T_\alpha(r, f') \sim \beta T_\alpha(r, f)$ and $\Delta_\alpha(0, f') = 1$.

Corollary 4.1 — If $\sum_{a \neq \infty} \delta_\alpha(a) = 1$ and $\delta_\alpha(\infty) = 1$,

then $\quad T_\alpha(r, f') \sim T_\alpha(r, f)$.

Theorem 5 — If $f(z)$ is a transcendental meromorphic function in $|z| < \infty$ with $\{a_i\}$ as modified deficient values, such that $\Sigma\, \delta_\alpha(a_i) = \beta (a_i \neq \infty)$ and $\delta_\alpha(\infty) = 2 - \beta$,

then $$\frac{\beta - 1}{\beta} \leqslant K_\alpha(f') \leqslant \frac{2(\beta - 1)}{\beta}$$

where $$K_\alpha(f') = \limsup_{r \to \infty} \frac{N_\alpha(r, f') + N_\alpha\left(r, \frac{1}{f'}\right)}{T_\alpha(r, f')}.$$

PROOF: Since $N(r, f') \leqslant 2N(r, f)$, we have

$$N_\alpha(r, f') \leqslant 2N_\alpha(r, f).$$

So $$\frac{N_\alpha(r, f')}{T_\alpha(r, f')} \frac{T_\alpha(r, f')}{T_\alpha(r, f)} \leqslant 2 \frac{N_\alpha(r, f)}{T_\alpha(r, f)}.$$

The equalities $\sum_{a_i \neq \infty} \delta_\alpha(a_i) = \beta$ and $\Sigma\, \delta_\alpha(a_i) = 2$

imply that $T_\alpha(r, f') \sim \beta T_\alpha(r, f)$.

Hence we have $\beta \limsup_{r \to \infty} \frac{N_\alpha(r, f')}{T_\alpha(r, f')} \leqslant 2(\beta - 1)$.

$$\therefore \quad \limsup_{r \to \infty} \frac{N_\alpha(r, f')}{T_\alpha(r, f')} \leqslant \frac{2(\beta - 1)}{\beta}.$$

Also we know that $\lim_{r \to \infty} \frac{N\left(r, \frac{1}{f'}\right)}{T(r, f')} = 0$.

Hence $$K_\alpha(f') \leqslant \frac{2(\beta - 1)}{\beta}. \qquad \ldots(5)$$

Further we have $N(r, f) \leqslant N(r, f')$,

hence $\quad N_\alpha(r, f) \leqslant N_\alpha(r, f')$.

$$\therefore \quad \frac{N_\alpha(r, f)}{T_\alpha(r, f)} \leqslant \beta \frac{N_\alpha(r, f')}{T_\alpha(r, f')}.$$

Hence $$\limsup_{r \to \infty} \frac{N_\alpha(r, f')}{T_\alpha(r, f')} \geqslant \frac{\beta - 1}{\beta}$$

$$\therefore \quad K_\alpha(f') \geqslant \frac{\beta - 1}{\beta}. \qquad \ldots(6)$$

Combining (5) and (6) we have

$$\frac{\beta - 1}{\beta} \leqslant K_\alpha(f') \leqslant \frac{2(\beta - 1)}{\beta}.$$

Theorem 6 — If $f(z)$ is a meromorphic function with $\{a_i\}$ where $(a_i \neq \infty)$ as modified deficient values such that $\Sigma\, \delta_\alpha(a_i) = \beta$ and $\mu_\alpha(\infty) = 2 - \beta$, then ∞ is not a modified deficient value of $f'(z)$, where

$$\mu_\alpha(a) = \liminf_{r \to \infty} \frac{N_\alpha(r, a) - \bar{N}_\alpha(r, a)}{T_\alpha(r, f)}.$$

Proof follows by usual techniques using second fundamental theorem and Theorem 4.

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