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SOME IDENTITIES INVOLVING THE RIEMANN ZETA FUNCTION

R. SITA RAMA CHANDRA RAO AND A. SIVA RAMA SARMA

Department of Mathematics, Andhra University, Waltair 530003

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In this paper, the authors discuss six identities involving the Riemann's zeta function two of which are respectively due to Williams (cf. Klamkin 1955, p. 130) and Gupta (1964-65) and another contains as a special case an identity due to Hans and Dumir (1964-65). Their arguments reduce the proofs of these six identities to proving just two of them. A proof of these two identities, based on a generalization of a transformation formula due to Lehner and Newman (1968-69) is given.

1. INTRODUCTION

Let a be an integer $\geqslant 2$ and $\zeta(s)$ be the Riemann's zeta function defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $s > 1$. In this paper we exhibit certain relationships among the following six identities, reduce the proofs of these identities to proving just two of them and finally prove them using essentially the same technique.

$$2 \sum_{r=1}^{\infty} \frac{1}{r^a} \sum_{k=1}^{r} \frac{1}{k} = (a+2)\,\zeta(a+1) - \sum_{i=1}^{a-2} \zeta(a-i)\,\zeta(i+1) \qquad \text{...(1.1)}$$

$$2 \sum_{r=1}^{\infty} \frac{1}{r^a} \sum_{\substack{k=1\\(k,r)=1}}^{r} \frac{1}{k} = a + 2 - \frac{1}{\zeta(a+1)} \sum_{i=1}^{a-2} \zeta(a-i)\,\zeta(i+1) \qquad \text{...(1.1}a\text{)}$$

$$\sum_{r=1}^{\infty} \sum_{k=1}^{r} \frac{1}{kr(k+r)} = \tfrac{5}{4}\,\zeta(3) \qquad \text{...(1.2)}$$

$$\sum_{r=1}^{\infty} \sum_{\substack{k=1\\(k,r)=1}}^{r} \frac{1}{kr(k+r)} = \tfrac{5}{4} \qquad \text{...(1.2}a\text{)}$$

$$\sum_{r=1}^{\infty} \sum_{k=1}^{r} \frac{1}{r^2(r+k)} = \tfrac{3}{4}\,\zeta(3) \qquad \text{...(1.3)}$$

$$\sum_{r=1}^{\infty} \sum_{\substack{k=1 \\ (k,r)=1}}^{r} \frac{1}{r^2(r+k)} = \frac{3}{4}. \qquad \ldots(1.3a)$$

The identity (1.1) is originally due to Williams (cf. Klamkin 1955, p. 130) and in case $a = 2$ reduces to a result due to Briggs *et al.* (1955). The special case $a = 3$ appears as a problem proposed by Klamkin (1953). (1.1*a*) in case $a = 2$ reduces to a result due to Hans and Dumir (1964-65) while (1.3*a*) is due to Gupta (1964-65). Our method of proof makes use of a generalization (Lemma 1 below) of a transformation formula due to Lehner and Newman (1968-69, Theorem 1) given as an exercise in Apostol (1976, p. 111) and a certain reciprocity relation (see Lemma 3 below) for the sums $\sum_{r=1}^{\infty} r^{-a} \sum_{k=1}^{r} k^{-b}$ where a, b are integers $\geqslant 2$.

2. Preliminaries

In this section we prove some results needed in the present discussion.

Lemma 1 (*A transformation formula*) — Let $f(x, y)$ be a real valued function defined for positive integral x and y. For a fixed positive integer n let T_n be the set of all ordered pairs (r, k) of positive integers such that $1 \leqslant r,\ k \leqslant n;\ r + k \geqslant n + 1$ and $S_n = \sum_{(x,y)\in T_n} f(x, y)$, the sum being taken over all pairs $(x, y) \in T_n$. Then

$$S_n = \sum_{r=1}^{n} f(r, r) + \sum_{r=2}^{n} \sum_{k=1}^{r-1} \{f(k, r) + f(r, k) - f(k, r-k)\}. \qquad \ldots(2.1)$$

Proof : When $n = 1$, this is trivial. When $n \geqslant 2$, we observe that T_n consists of all pairs (x, y) of positive integers lying in the closure of the triangle bounded by the lines $x + y = n + 1$, $x = n$ and $y = n$. Hence for $r \geqslant 2$, the points of T_r not in T_{r-1} are all the pairs (x, y) of integers lying on the line segments $x = r$, $1 \leqslant y \leqslant r$; $y = r$, $1 \leqslant x \leqslant r$ and the points of T_{r-1} not in T_r are all the pairs (x, y) of integers lying on the line segment $x + y = r$, $1 \leqslant x, y \leqslant r$. This implies that for $r \geqslant 2$

$$S_r - S_{r-1} = \sum_{k=1}^{r-1} \{f(k, r) + f(r, k) - f(k, r-k)\} + f(r, r).$$

Taking $r = 2, 3, \ldots, n$ in the above and summing the resulting equations, the lemma follows.

Remark 1 : In case $f(x, y)$ satisfies $f(x, y) = 0$ whenever the g.c.d. $(x, y) > 1$, we note that Lemma 1 reduces to a transformation formula due to Lehner and Newman (1968-69, Theorem 1).

Lemma 2 — Let $f(x, y)$ be as in Lemma 1 and be homogeneous of order $\alpha < -1$, i.e. for each positive integer t, $f(tx, ty) = t^{\alpha} f(x, y)$. Then the series $\sum_{r=1}^{\infty} \sum_{k=1}^{r} f(k, r)$ converges iff the series $\sum_{r=1}^{\infty} \sum_{\substack{k=1 \\ (k,r)=1}}^{r} f(k, r)$ converges and in case of convergence

$$\sum_{r=1}^{\infty} \sum_{k=1}^{r} f(k, r) = \zeta(-\alpha) \sum_{r=1}^{\infty} \sum_{\substack{k=1 \\ (k,r)=1}}^{r} f(k, r).$$

PROOF: We write $a_r = \mu(r)\, r^{\alpha}$, $b_r = \sum_{k=1}^{r} f(k, r)$ and $c_r = \sum_{\substack{k=1 \\ (k,r)=1}}^{r} f(k, r)$ where μ is the Möbius function. Then

$$\sum_{d \mid r} a_d b_{r/d} = \sum_{d \mid r} \mu(d) \sum_{k=1}^{r/d} d^{\alpha} f\left(k, \frac{r}{d}\right) = \sum_{d \mid r} \mu(d) \sum_{\substack{k=1 \\ d \mid k}}^{r} f(k, r)$$

$$= \sum_{k=1}^{r} f(k, r) \left(\sum_{d \mid r, d \mid k} \mu(d) \right) = \sum_{\substack{k=1 \\ (k,r)=1}}^{r} f(k, r) = c_r$$

where we used the fact that $\sum_{d \mid n} \mu(d) = 1$ or 0 according as $n = 1$ or $n > 1$. Similarly writing $a'_r = r^{\alpha}$ we see that $\sum_{d \mid r} a'_d\, c_{r/d} = b_r$. Now it is known that if the series $\sum_{n=1}^{\infty} \alpha_n$ converges absolutely and the series $\sum_{n=1}^{\infty} \beta_n$ converges, then the Dirichlet product series $\sum_{n=1}^{\infty} \nu_n$ converges where $\nu_n = \sum_{d \mid n} \alpha_d \beta_{n/d}$ and further

$$\sum_{n=1}^{\infty} \nu_n = \left(\sum_{n=1}^{\infty} \alpha_n \right) \left(\sum_{n=1}^{\infty} \beta_n \right)$$

(Landau 1953, section 185, p. 671). Since the series $\sum_{n=1}^{\infty} n^{\alpha}$ and $\sum_{n=1}^{\infty} \mu(n)\, n^{\alpha}$ both converge absolutely and $\sum_{n=1}^{\infty} \mu(n)\, n^{\alpha} = (1/\zeta(-\alpha))$, the Lemma follows.

Lemma 3 (*Reciprocity relations*) — Let a, b be integers $\geqslant 2$ and write

$$P(a, b) = \sum_{r=1}^{\infty} r^{-a} \sum_{k=1}^{r} k^{-b} \quad \text{and} \quad Q(a, b) = \sum_{r=1}^{\infty} r^{-a} \sum_{\substack{k=1 \\ (k, r)=1}}^{r} k^{-b}.$$

Then

$$P(a, b) + P(b, a) = \zeta(a)\,\zeta(b) + \zeta(a + b) \qquad \ldots(2.2)$$

$$Q(a, b) + Q(b, a) = \frac{\zeta(a)\,\zeta(b)}{\zeta(a + b)} + 1. \qquad \ldots(2.3)$$

PROOF : We have

$$\zeta(a)\,\zeta(b) = \sum_{k, r=1}^{\infty} \frac{1}{r^a k^b} = \sum_{r=1}^{\infty} \frac{1}{r^a} \sum_{k \leqslant r} \frac{1}{k^b} + \sum_{r=1}^{\infty} \frac{1}{r^a} \sum_{k > r} \frac{1}{k^b}$$

$$= P(a, b) + \sum_{k=2}^{\infty} \frac{1}{k^b} \sum_{r=1}^{k-1} \frac{1}{r^a}$$

$$= P(a, b) + \sum_{k=1}^{\infty} \frac{1}{k^b} \sum_{r=1}^{k} \frac{1}{r^a} - \sum_{k=1}^{\infty} \frac{1}{k^{b+a}}$$

$$= P(a, b) + P(b, a) - \zeta(a + b)$$

which proves (2.2).

To prove (2.3) we note that by Lemma 2

$$Q(a, b) + Q(b, a) = \frac{1}{\zeta(a + b)} \{P(a, b) + P(b, a)\}.$$

3. MAIN RESULTS

In this section, we give proofs of the six identities (1.1) through (1.3*a*). First we note that, by Lemma 2, each of the identities (1.1), (1.2) and (1.3) is respectively equivalent to the identities (1.1*a*), (1.2*a*) and (1.3*a*). Also, since

$$\frac{1}{r^2 k} - \frac{1}{kr(k + r)} = \frac{1}{r^2(k + r)},$$

we see that any two of the identities namely, (1.1) in case $a = 2$, (1.2) and (1.3) [Also (1.1*a*) in case $a = 2$, (1.2*a*) and (1.3*a*)] imply the third. Hence it is sufficient to prove (1.1) and (1.2).

To prove (1.1), we take $f(x, y) = \dfrac{1}{x^a y}$ in the transformation formula (2.1) and obtain

$$S_n = \sum_{r=1}^{n} \frac{1}{r^{a+1}} + \sum_{r=2}^{n} \sum_{k=1}^{r-1} \left\{ \frac{1}{k^a r} + \frac{1}{r^a k} - \frac{1}{k^a(r - k)} \right\}$$

(*equation continued on p.* 606)

$$= a \sum_{r=1}^{n} \frac{1}{r^{a+1}} - \sum_{r=1}^{n} \sum_{k=1}^{r} \left\{ \frac{r^{a-2} + r^{a-3}k + \ldots + rk^{a-3} + k^{a-2}}{k^{a-1}r^{a}} \right\} \quad \ldots(3.1)$$

by a straightforward calculation. Also, by definition, for any integer $a \geqslant 2$ we have

$$S_n = \sum_{\substack{1 \leqslant x, y \leqslant n \\ x+y \geqslant n+1}} \frac{1}{x^a y} \leqslant \sum_{\substack{1 \leqslant x, y \leqslant n \\ x+y \geqslant n+1}} \frac{1}{x^2 y} \leqslant \sum_{y \leqslant n} \frac{1}{y} \sum_{x \geqslant n+1-y} \frac{1}{x^2}$$

$$= O\Big(\sum_{y \leqslant n} \frac{1}{y(n+1-y)} \Big) = O\Big(\frac{1}{n+1} \sum_{y \leqslant n} \Big\{ \frac{1}{y} + \frac{1}{n+1-y} \Big\} \Big)$$

$$= O\Big(\frac{1}{n+1} \sum_{y \leqslant n} \frac{1}{y} \Big)$$

$$= O\Big(\frac{\log 2n}{n} \Big). \quad \ldots(3.2)$$

Taking limits as $n \to \infty$ on both sides of (3.1) while using (3.2), we get

$$a\zeta(a+1) = \sum_{r=1}^{\infty} \sum_{k=1}^{r} \frac{1}{r^a k} + \sum_{i=1}^{a-2} P(a-i, i+1).$$

Now we use (2.2) of Lemma 3 to obtain

$$2a\zeta(a+1) = 2 \sum_{r=1}^{\infty} \sum_{k=1}^{r} \frac{1}{r^a k} + \sum_{i=1}^{a-2} \{ P(a-i, i+1) + P(i+1, a-i) \}$$

$$= 2 \sum_{r=1}^{\infty} \sum_{k=1}^{r} \frac{1}{r^a k} + \sum_{i=1}^{a-2} \{ \zeta(a-i)\, \zeta(i+1) + \zeta(a+1) \}$$

from which (1.1) follows.

To prove (1.2), we take $f(x, y) = \dfrac{2y + x}{2x^2 y(x+y)}$ in the transformation formula (2.1) and after some simplification, obtain

$$S'_n = \sum_{(x,y) \in T_n} f(x, y) = \sum_{r=1}^{n} \frac{3}{4r^3} - \sum_{r=2}^{n} \sum_{k=1}^{r-1} \left\{ \frac{r^2 k - k^2 r + 2k^3}{2r^2 k^2 (r^2 - k^2)} \right\}$$

$$= \frac{3}{4} \sum_{r=1}^{n} \frac{1}{r^3} - \sum_{r=2}^{n} \sum_{k=1}^{r-1} \frac{1}{rk(r+k)} \quad \ldots(3.3)$$

since for any integer $r \geqslant 2$

$$2\sum_{k=1}^{r-1}\left\{\frac{r^2k - k^2r + 2k^3}{2r^2k^2(r^2 - k^2)}\right\}$$

$$= \sum_{k=1}^{r-1}\left\{\frac{r^2k - k^2r + 2k^3}{2r^2k^2(r^2 - k^2)} + \frac{r^2(r-k) - r(r-k)^2 + 2(r-k)^3}{2r^2(r-k)^2\, k(2r-k)}\right\}$$

$$= \sum_{k=1}^{r-1}\left\{\frac{1}{rk(r+k)} + \frac{1}{r(r-k)\,(2r-k)}\right\} = 2\sum_{k=1}^{r-1}\frac{1}{kr(k+r)}.$$

Also, by (3.2), we have

$$S'_n \leqslant 2S_n = O\left(\frac{\log 2n}{n}\right).$$

Now taking limits as $n \to \infty$ on both sides of (3.3) we obtain

$$\tfrac{3}{4}\,\zeta(3) = \sum_{r=2}^{\infty}\left(\sum_{k=1}^{r-1}\frac{1}{rk(r+k)}\right)$$

which leads at once to (1.2).

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