Indian J. pure appl. Math., **10**(5) : 608-616, May 1979

SOME SOLUTIONS OF SPHERICAL CHARGED FLUID BALLS

SUBHENDU P. CHAKRAVARTI* AND UTPAL K. DE

Physics Department, Jadavpur University, Calcutta 700032

(*Received* 20 *November* 1978)

The authors have presented some new solutions of spherical charged fluid ball, which are regular everywhere and can be matched with outside Reissner-Nordström solution. In all the solutions, the ratio of the total charge to total mass of the ball is less than unity in relativistic units ($C = G = 1$).

1. INTRODUCTION

The study of statics and dynamics of charged matter distributions in General Relativity has attracted the attention of many workers. It is now a known fact that a pressureless charged dust distribution in equilibrium will have the absolute value of the charge to mass ratio as unity in relativistic units i.e. $C = G = 1$ (De and Raychaudhuri 1968). A number of authors have already studied the charged fluid distribution in equilibrium. While Efinger (1965), Bailyn and Eimerl (1972) and recently Nduka (1977) have presented some solutions of static spherical distribution which are not free from singularity at the origin, Omote (1973) has given a solution where the charge to mass ratio of the spherical ball can be made arbitrarily large. However, because of the repulsive action of the pressure, the charge to mass ratio of a spherical ball is expected to be less than unity in case of fluid matter as the ratio becomes unity for pressureless dust matter. This strange result was obtained by Omote (1973) possibly because of an unexplained hyper-geometric function appearing in his solution. Incidentally, there also exist some solutions for charged fluid distributions with spherical symmetry which are very regular (Kyle and Martin 1967, Wilson 1967, Kramar and Neugebauer 1971, Krori and Barua 1975). In this paper, the authors also study the problem of static charged fluid distributions in the form of a sphericall ball and present some new solutions which are regular eveywhere and of course, the solutions could be matched with outside Reissner-Norström metric. In all our solutions, the charge to mass ratio of the spherical ball is as expected, less than unity.

In section 2 of the paper, the field equations and the conditions to be satisfied by the metric components to have a regular solution are written down, whereas the solutions are presented in section 3. Some features of the solutions are discussed in section 4.

---

*On leave from Physics Department, B. N. Mahavidyalaya, Itachuna, Hooghly, West Bengal, India.

2. The Field Equations and the Conditions

The static spherically symmetric metric may be written as

$$dS^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \quad ...(2.1)$$

The Einstein-Maxwell equations for a charged fluid matter of mass density ρ, charge density σ and pressure P and having a radial electric field $F^{\mu\nu}$ are

$$8\pi\rho + 8\pi E_0^0 = e^{-\lambda}\left[\frac{\lambda'}{r} - \frac{1}{r^2}\right] + \frac{1}{r^2} \quad ...(2.2a)$$

$$8\pi P - 8\pi E_1^1 = e^{-\lambda}\left[\frac{\nu'}{r} + \frac{1}{r^2}\right] - \frac{1}{r^2} \quad ...(2.2b)$$

$$8\pi P - 8\pi E_2^2 = e^{-\lambda}\left[\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\lambda'\nu'}{4} + \frac{\nu' - \lambda'}{2r}\right] \quad ...(2.2c)$$

$$F'_{10} + F_{10}\left(\frac{2}{r} - \frac{\lambda' + \nu'}{2}\right) = 4\pi\sigma e^{(\nu+2\lambda)/2} \quad ...(2.2d)$$

where $8\pi E^2 = - F_{10}F^{10}$ and the prime indicates differentiation with respect to radial coordinate. The suffixes 0 through 3 are respectively for t, r, θ and φ directions.

If the metric components, the matter density, the electric field energy and the pressure are to be regular and finite around the origin $r = 0$, the following conditions must be satisfied :

(i) $\underset{r\to 0}{\mathrm{Lt}}\,(e^{\lambda/2}) = 1$

(ii) $\underset{r\to 0}{\mathrm{Lt}}\,(E^2) = 0$

(iii) From eqns. (2.2a) and (2.2b), we find that $\underset{r\to 0}{\mathrm{Lt}}\,(\nu') \propto r^\alpha$

and $\underset{r\to 0}{\mathrm{Lt}}\,(\lambda') \propto r^\beta$, where α and β must be equal to or greater than one. This is necessary to keep the matter density and the pressure finite at the origin.

In order to avoid the singularity at the origin we will have to choose e^λ and e^ν satisfying the above three conditions. Let us choose,

$$e^\lambda = (1 + Ar^2)^m \quad ...(2.3a)$$

and

$$e^\nu = C(1 + Br^2)^n \quad ...(2.3b)$$

where A, B, C, m and n are finite constant quantities to be determined from other conditions. However C may be removed by the transformation of t-coordinate but we retain it for the purpose of matching the inside metric with outside Reissner-Nordström metric.

Equations (2.2a), (2.2b) and (2.2c) may now be rewritten as

$$16\pi\rho = \frac{1}{(1+Ar^2)^m}\left[\frac{5Am}{(1+Ar^2)} + r^2\left\{\frac{2B^2n(1+Ar^2) - 2B^2n^2(1+Ar^2) + ABmn(1+Br^2)}{(1+Ar^2)(1+Br^2)^2}\right\} - \frac{1}{r^2}\right] + \frac{1}{r^2} \quad \ldots(2.4a)$$

$$16\pi P = \frac{1}{(1+Ar^2)^m}\left[\frac{4Bn}{(1+Br^2)} - \frac{Am}{(1+Ar^2)} - r^2\left\{\frac{2B^2n(1+Ar^2) - B^2n^2(1+Ar^2) + ABmn(1+Br^2)}{(1+Ar^2)(1+Br^2)^2}\right\} + \frac{1}{r^2}\right] - \frac{1}{r^2} \quad \ldots(2.4b)$$

$$16\pi E^2 = \frac{1}{(1+Ar^2)^m}\left[r^2\left\{\frac{B^2n^2(1+Ar^2) - 2B^2n(1+Ar^2) - ABmn(1+Ar^2)}{(1+Br^2)^2(1+Ar^2)}\right\} - \frac{Am}{(1+Ar^2)} - \frac{1}{r^2}\right] + \frac{1}{r^2} \quad \ldots(2.4c)$$

A little examination shows that eqns. (2.3a) and (2.3b) satisfy all the three conditions stated previously and

$$\operatorname*{Lt}_{r\to 0} 16\pi\rho = 6Am$$

and

$$\operatorname*{Lt}_{r\to 0} 16\pi P = 2(2nB - mA).$$

Since the pressure and matter density must be positive and the ratio P/ρ is $\leqslant \frac{1}{3}$, we find that

$$2Bn > mA \geqslant Bn \quad \ldots(2.5a)$$

and

$$Bn > 0. \quad \ldots(2.5b)$$

Obviously if we want to consider any ratio between A and B, and between m and n, we will have to remember (2.5a) and (2.5b).

3. The Solutions

In eqns. (2.3a) and (2.3b), we consider a simple situation when $A = B$. Then (2.5a) and (2.5b) may be combined and rewritten as

$$| 2n | > | m | \geqslant | n | . \qquad \ldots(3.1)$$

Now one can put various positive and negative values of m and the value of n can be picked up satisfying the condition (3.1). However, for a regular solution the pressure, the matter density and the electric energy density must be positive everywhere and P/ρ must also be $\leqslant \frac{1}{3}$ everywhere. Besides, if we want to bound the distribution, the pressure must vanish at a certain value of the radial coordinate where the boundary will exist and where the inside metric must be matched with outside Reissner-Nordström metric.

Condition (3.1) has been obtained considering the pressure and the ratio of density to pressure around the origin $r = 0$. Now we will consider the behaviour of E^2 as $r \rightarrow 0$. Putting $A = B$, eqn. (2.4c) may be reduced to

$$16\pi E^2 = \frac{\left(n^2 - 2n - mn + \frac{m^2}{2} + \frac{m}{2}\right)A^2r^2 + {}^{(m+2)}C_3(A^3r^4) + {}^{(m+2)}C_4(A^4r^6) + \ldots + A^{m+2}\,.\,r^{2m+2}}{(1 + Ar^2)^{m+2}} \qquad \ldots(3.2)$$

As $r \rightarrow 0$, the term

$$\left(n^2 - 2n - mn + \frac{m^2}{2} + \frac{m}{2}\right)A^2r^2 \qquad \ldots(3.3)$$

becomes most dominant in the numerator of eqn. (3.2). So, in order to keep E^2 positive as $r \rightarrow 0$, the above term must be positive.

We consider here only the positive values of m. One can check that for $m = 1$ and 2, the permissible values of n which keep eqn. (3.3) positive, violate eqn. (3.1). When $m = 3$, eqn. (3.3) becomes $[(n - 2)(n - 3)]\,A^2r^2$ and the permissible value of n keeping E^2 positive and satisfying eqn. (3.1) are $n = 3$ and $2 \geqslant n > \frac{3}{2}$. For higher values of m, E^2 remains positive throughout, and the only restriction on n comes from eqn. (3.1).

Thus, we have examined fully the regularity of the solutions around the origin $r = 0$. However all the solutions indicated above may not be solutions for bounded charged mass distribution unless those could be matched with outside Reissner-Nordström solution. In the following we shall present only three particular solutions which could be matched with outside Reissner-Nordström solution.

Solution 1 : $m = 3$, $n = 2$

Then

$$e^{\lambda} = (1 + Ar^2)^3 \qquad \ldots(3.4a)$$

$$e^{\nu} = C(1 + Ar^2)^2 \qquad \ldots(3.4b)$$

$$16\pi\rho = \frac{18A + 30A^2r^2 + 10A^3r^4 + 5A^4r^6 + A^5r^8}{(1 + Ar^2)^5} \qquad ...(3.4c)$$

$$16\pi P = \frac{2A - 10A^2r^2 - 10A^3r^4 - 5A^4r^6 - A^5r^8}{(1 + Ar^2)^5} \qquad ...(3.4d)$$

$$16\pi E^2 = \frac{10A^3r^4 + 5A^4r^6 + A^5r^8}{(1 + Ar^2)^5}. \qquad ...(3.4e)$$

Solution 2 : $m = n = 3$

Then

$$e^\lambda = (1 + Ar^2)^3 \qquad ...(3.5a)$$

$$e^\nu = C(1 + Ar^2)^3 \qquad ...(3.5b)$$

$$16\pi\rho = \frac{18A + 30A^2r^2 + 10A^3r^4 + 5A^4r^6 + A^5r^8}{(1 + Ar^2)^5} \qquad ...(3.5c)$$

$$16\pi P = \frac{6A - 6A^2r^2 - 10A^3r^4 - 5A^4r^6 - A^5r^8}{(1 + Ar^2)^5} \qquad ...(3.5d)$$

$$16\pi E^2 = \frac{A^5r^8 + 5A^4r^6 + 10A^3r^4}{(1 + Ar^2)^5}. \qquad ...(3.5e)$$

Solution 3 : $m = 4, \quad n = 3$

Then

$$e^\lambda = (1 + Ar^2)^4 \qquad ...(3.6a)$$

$$e^\nu = C(1 + Ar^2)^3 \qquad ...(3.6b)$$

$$16\pi\rho = \frac{24A + 43A^2r^2 + 20A^3r^4 + 15A^4r^6 + 6A^5r^8 + A^6r^{10}}{(1 + Ar^2)^6} \qquad ...(3.6c)$$

$$16\pi P = \frac{4A - 15A^2r^2 - 20A^3r^4 - 15A^4r^6 - 6A^5r^8 - A^6r^{10}}{(1 + Ar^2)^6}. \qquad ...(3.6d)$$

$$16\pi E^2 = \frac{A^2r^2 + 20A^3r^4 + 15A^4r^8 + 6A^5r^8 + A^6r^{10}}{(1 + Ar^2)^6}. \qquad ...(3.6e)$$

4. Matching with Outside Metric, Mass Defect and Discussions

When the matching of the outside Reissner-Nordström metric with the inside metric is considered, all the metric components must be continuous across the boundary. Further, across the boundary both the pressure and the electric field must be continuous, though there will remain a discontinuity in density. Hence it is obvious from eqn. (2.2*b*) that ν' of eqn. (2.1) must also remain continuous across the boundary.

Now, let the boundary exist at $r = R_0$, then we will have the following equations :

$$(e^{\nu})_{\text{at } r=R_0} = C(1 + AR_0^2)^n = \left(1 - \frac{2M}{R_0} + \frac{e^2}{R_0^2}\right) \quad \text{...(4.1a)}$$

$$(e^{-\lambda})_{\text{at } r=R_0} = (1 + AR_0^2)^{-m} = \left(1 - \frac{2M}{R_0} + \frac{e^2}{R_0^2}\right) \quad \text{...(4.1b)}$$

$$(\nu' e^{\nu})_{\text{at } r=R_0} = 2AnR_0C(1 + AR_0^2)^{n-1}$$

$$= \left(\frac{2M}{R_0^2} - \frac{2e^2}{R_0^3}\right) \quad \text{...(4.1c)}$$

where M and e are the total mass and charge of the distribution. Solving the above three equations, one finds that

$$C = (1 + AR_0^2)^{-(m+n)} \quad \text{...(4.2)}$$

$$\frac{e^2}{R_0^2} = 1 - [1 + (2n + 1)AR_0^2][1 + AR_0^2]^{-(m+1)} \quad \text{...(4.3)}$$

$$\frac{M}{R_0} = 1 - [1 + (n + 1)AR_0^2][1 + AR_0^2]^{-(n+1)}. \quad \text{...(4.4)}$$

Since both M and e^2 must be positive, the restriction on R_0 comes through the equation

$$(1 + AR_0^2)^{m+1} > [1 + (2n + 1)AR_0^2]. \quad \text{...(4.5)}$$

The value of R_0 found by putting $P = 0$ in eqn. (2.4b), must satisfy (4.5).

The mass-defect may be calculated from the difference of M and the mass calculated from the integration of density over the spatial volume of the distribution. If M_0 is the later mass, then

$$M_0 = \int_\tau \sqrt{-g} \cdot \rho(r)\, v^0\, d\tau$$

where v^0 is the time-component of the 4-velocity and τ the spatial volume.

Then

$$M_0 = \tfrac{1}{4} \int_0^{R_0} [5Amr^2(1 + Ar^2)^{-(m+2)/2} + (2n - n^2 + mn)$$

$$\times A^2r^4(1 + Ar^2)^{-(m+4)/2} - (1 + Ar^2)^{-m/2} + (1 + Ar^2)^{m/2}]\, dr. \quad \text{...(4.6)}$$

The mass M, as observed by an outside observer may be written as

$$M = \frac{e^2}{R_0} + AnR_0^3(1 + AR_0^2)^{-(m+1)} \quad \text{...(4.7)}$$

and the mass-defect

$$\delta M = M - M_0. \qquad \ldots(4.8)$$

The integrations of eqn. (4.6) may be performed following any standard table (Dwight 1961).

Lastly the expression for the square of the ratio of total charge to total mass is given by

$$\left(\frac{e}{M}\right)^2 = \left[1 - \frac{1 + (1 + 2n)\, AR_0^2}{(1 + AR_0^2)^{m+1}}\right]\left[1 - \frac{1 + (1 + n)\, AR_0^2}{(1 + AR_0^2)^{m+1}}\right]^{-2} \qquad \ldots(4.9)$$

and if M^2 is to be greater than e^2, the following condition must be satisfied :

$$[1 + (n + 1)\, AR_0^2]^2 > (1 + AR_0^2)^{m+2} \qquad \ldots(4.10)$$

Now let us consider the matching at the boundary, the value of mass-defect and the possibility of satifying (4.10) in the three solutions separately.

Solution 1 : $m = 3, n = 2$

$$e^{\lambda} = (1 + Ar^2)^3 \quad \text{and} \quad e^{\nu} = (1 + AR_0^2)^{-5}\,(1 + Ar^2)^2$$

where R_0 is the radius of the boundary.

The boundary exists at R_0 such that,

$$2 - 10(AR_0^2) - 10(AR_0^2)^2 - 5(AR_0^2)^3 - (AR_0^2)^4 = 0 \qquad \ldots(4.11)$$

The real value of (AR_0^2) satisfying the above equation is approximately 1/5.89 and this value of (AR_0^2) satisfies the condition (4.5). So the outside Reissner-Nordström metric can be matched smoothly at the boundary.

From eqn. (4.8), mass defect

$$\begin{aligned}\delta M = \frac{e^2}{R_0} &+ 2AR_0^3\,(1 + AR_0^2)^{-4} - \frac{5}{4}\, AR_0^3\,(1 + AR_0^2)^{-3/2} \\ &- \frac{3}{10}\, A^2R_0^5\,(1 + AR_0^2)^{-5/2} + \frac{1}{4}\, R_0(1 + AR_0^2)^{-1/2} \\ &- \frac{1}{16}\, R_0(1 + AR_0^2)^{3/2} - \frac{3}{32}\, R_0(1 + AR_0^2)^{1/2} \\ &+ \frac{3}{32}\, A^{-1/2} \log\{R_0 + A^{-1/2}\,(1 + AR_0^2)\}.\end{aligned}$$

Also, with $AR_0^2 = 1/5.89$ and $m = 3$, $n = 2$, (4.10) is found to be satisfied i.e. the total mass of the distribution is larger than the total charge.

Solution 2 : $m = n = 3$

$$e^{\lambda} = (1 + Ar^2)^3, \; e^{\nu} = (1 + AR_0^2)^{-6} (1 + Ar^2)^3.$$

The boundary R_0 is obtained from

$$6 - 6(AR_0^2) - 10(AR_0^2)^2 - 5(AR_0^2)^3 - (AR_0^2)^3 - (AR_0^2)^4 = 0.$$

The real value of (AR_0^2) satisfying the above equation is approximately 1/2.01 and this value of (AR_0^2) satisfies (4.5). Hence the inside metric can be matched smoothly with the outside metric.

From eqn. (4.8), mass defect

$$\delta M = \frac{e^2}{R_0} + 3AR_0^3 (1 + AR_0^2)^{-4} - \frac{5}{4} AR_0^3 (1 + AR_0^2)^{-3/2}$$
$$- \frac{3}{10} A^2R_0^5 (1 + AR_0^2)^{-5/2} + \frac{1}{4} R_0(1 + AR_0^2)^{-1/2}$$
$$- \frac{1}{16} R_0(1 + AR_0^2)^{3/2} - \frac{3}{32} R_0(1 + AR_0^2)^{1/2}$$
$$+ \frac{3}{32} A^{-1/2} \log \{R_0 + A^{-1/2}(1 + AR_0^2)\}.$$

Also, with $AR_0^2 = 1/2.01$ and $m = n = 3$, (4.10) is satisfied, so that total mass of the distribution is greater than the total charge.

Solution 3 : $m = 4, \; n = 3$

$$e^{\lambda} = (1 + Ar^2)^4, \; e^{\nu} = (1 + AR_0^2)^{-7} (1 + Ar^2)^3.$$

The boundary R_0 may be obtained from the equation

$$4 - 15(AR_0^2) - 20(AR_0^2)^2 - 15(AR_0^2)^3 - 6(AR_0^2)^4 - (AR_0^2)^5 = 0.$$

The real value of (AR_0^2) satisfying the above equation is approximately 1/4.93 and this value of (AR_0^2) satisfies again (4.5). Hence the inside metric can be matched with outside Reissner-Nordström metric here also.

From eqn. (4.8), mass defect

$$\delta M = \frac{e^2}{R_0} + 3AR_0^3 (1 + AR_0^2)^{-5} - \frac{3}{8} R_0(1 + AR_0^2)^{-3}$$
$$+ \frac{61}{32} R_0(1 + AR_0^2)^{-2} - \frac{41}{64} R_0(1 + AR_0^2)^{-1}$$
$$- \frac{1}{4} R_0 - \frac{1}{6} AR_0^3 - \frac{1}{20} A^2R_0^5 - \frac{41}{64} \cdot A^{-1/2} \tan^{-1} (A^{1/2}R_0).$$

Again with $AR_0^2 = 1/4.93$ and $m = 4$, $n = 3$, (4.10) is found to be valid and hence the mass of the distribution remains greater than the total charge amount.

It is obvious that following our procedure one can construct an innummerable number of solutions for spherical charged fluid balls which are regular everywhere and which can also be matched with outside Reissner-Nordström solution.

Acknowledgement

One of the authors (S.P.C.) wishes to express thanks to U.G.C., New Delhi, for a teacher fellowship.

References

Bailyn, M., and Eimerl, D. (1972). General-relativistic interior metric for a stable static charged matter fluid with large *e/m*. *Phys. Rev.*, D **5**, 1897-1907.

De, U. K., and Raychaudhuri, A. K. (1968). Static distribution of charged dust in general relativity. *Proc. R. Soc.*, A **303**, 97-101.

Dwight, H. B. (1961). Tables of Integrals and other Mathematical Data. The Macmillan Company, New York.

Efinger, H. J. (1965). Uber die Selbstenergie und Ladurg eines durch Gravitationswirkungen Stabilisierten Teilchens in einem gekrummten Raum. *Z. Phys.*, **188**, 31-37.

Kramer, D., and Neugebauer, G. (1971). *Ann. Phys. Lpz.*, **27**, 129.

Krori, K. D., and Barua, J. (1975). A singularity-free solution for a charged fluid sphere in general relativity. *J. Phys.*, A **8**, 508-11.

Kyle, C. F., and Martin, A. W. (1967). Self-energy considerations in general relativity and the exact fields of charge and mass distributions. *Il Nuovo Cim.*, **50**, 583-604.

Nduka, A. (1977). Some exact solutions of charged general relativistic fluid spheres. *Acta Phys. Pol.*, B **8**, 75-79.

Omote, M. (1973). General-relativistic interior metric for a static charged fluid ball with pressure. *Lett. Nuovo Cim.*, **6**, 49-52.

Wilson, S. J. (1967). Exact solution of a static charged sphere in general relativity. *Can. J. Phys.*, **47**, 2401-2404.