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INTEGRATION OF VECTOR-VALUED FUNCTIONS

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Let X, E and F be linear lattices over reals; let $(S, Q(S), \iota, X)$ be a measure space, where S is a non-atomic complete Boolean Algebra, $Q(S)$ a σ-algebra of subsets of S and ι a X-valued measure. With respect to this measure space, the authors have discussed a theory of integration for a class of functions $f : S \rightarrow E \cup \{\infty\}$ in such a manner as to ensure that the values of the integrals so formed lie in F.

1. INTRODUCTION

The purpose of the present paper is to develop a theory of integration for a class of linear-lattice-valued functions with respect to measures on sets which take values in another linear lattice. To this end, the function ι constructed in (Kundu and Lahiri 1978) on the power set $P(S)$ of a non-atomic complete boolean algebra S suits quite well because of its measure-theoretic properties. The major departure in the present paper from the papers where integration theory was developed under similar situations (see the papers in the list of references) is that they define integration with respect to measures under axioms, while we do so with respect to measures that we constructed. Another point of variation is that here the domain space or the range space of measures and functions are both abstract spaces i.e. either they are boolean algebra or lattices under certain restrictions.

Here we have freely used properties of linear lattices e.g. order convergence in short 0-convergence or its obvious variations without reference to each situation, because either they occur in standard texts (see Birkhoff 1967, Nakano 1950, Peressini 1967) or they follow from known definitions. We shall adjoin an object ∞ to a linear lattice L and shall fit it with the ordering in L by the rule $a < \infty$ for every $a \in L$. L^+ will denote the set $\{a \in L \mid a \geqslant 0\}$ and a Nc b will mean a is not comparable with b.

2. BASIC DEFINITIONS

Let S be a non-atomic complete boolean algebra, H a boolean subalgebra of S and X be a conditionally complete lattice-ordered group.

Definition 2.1 — For $a \in S$, let $H^{+}(a) = \{\alpha \in H \mid a \leqslant \alpha\}$, and

$$H^{-}(a) = \{\alpha \in H \mid \alpha \leqslant a\}.$$

Definition 2.2 — Let $f : H \to X$ be such that

(a) $x \leqslant y \Rightarrow f(x) \leqslant f(y)$;

(b) For every arbitrary sequence of elements $\{\alpha_i\}$ with $\bigvee_{i=1}^{\infty} \alpha_i$ in H, $f(\bigvee_{i=1}^{\infty} \alpha_i) \leqslant \sum_{i=1}^{\infty} f(\alpha_i)$, provided the sum exists; equality holds if the elements form a disjoint sequence;

(c) For an increasing (decreasing) sequence of elements $\{\alpha_i\}$ in H with $\alpha = \bigvee_{i=1}^{\infty} \alpha_i$ $(\alpha = \bigwedge_{i=1}^{\infty} \alpha_i)$ in H, $f(\alpha) = \bigvee_{i=1}^{\infty} f(\alpha_i)$ $(f(\alpha) = \bigwedge_{i=1}^{\infty} f(\alpha_i))$;

(d) $f(0) = 0$.

Definition 2.3 — For $a \in S$, let $g = \sup_{\alpha \in H^{-}(a)} f(\alpha)$, and $\mu(a) = \inf_{\alpha \in H^{+}(a)} g(\alpha)$.

It was observed (Kundu 1972) that the extensions g and μ possess all the properties of inner and outer measure functions respectively.

Definition 2.4 — For $A \in P(S)$, let

$$\iota(A) = \mu(\sup A), \quad A \neq \phi \quad \text{and} \quad \iota(\phi) = 0.$$

Definition 2.5 — Let

$$Q(S) = \{A \in P(S) \mid \iota(T) = \iota(T \cap A) + \iota(T \cap A^{c})$$

for every $T \in P(S)$ where A^{c} is the complement of $A\}$.

It was shown Kundu and Lahiri (1978) that $Q(S)$ is a σ-subalgebra of $P(S)$ and that ι has all the properties of a measure function on $Q(S)$.

Definition 2.6 — The ordered pair $(S, Q(S))$ shall be called a measurable space and the members of $Q(S)$ shall be called measurable sets.

Definition 2.7 — The ordered set $(S, Q(S), \iota, X)$ shall be called a measure space.

Definition 2.8 — Let E be a linear lattice. A function $f: S \to E \cup \{\infty\}$ is said to be measurable with respect to $(S, Q(S))$ if for all $a \in E$, the sets $\{x : a < f(x)\}$ and $\{x : a$ Nc $f(x)\}$ are measurable.

Definition 2.9 — A lattice E is said to be operative if for every set $M \subset E$ with $\sup M$ and $\inf M$ in E the following hold :

(i) if $c < \sup M$, there exists at least one $x \in M$ such that $c < x$;

(ii) if $\inf M < c$, there exists at least one $x \in M$ such that $x < c$.

The set R of real numbers with the lattice structures of upper bound and lower bound is an operative lattice.

It may be noted that if $\{f_n\}$ is a sequence of measurable functions such that $f_n : S \to E \cup \{\infty\}$, where E is an operative lattice, then $0 - \underline{\lim}_n f_n$ and $0 - \overline{\lim}_n f_n$ are measurable, provided they exist.

The symbols E, F etc. will stand for linear lattices over reals and restrictions, if there be any, will be mentioned at the appropriate places.

3. Bilinear Function (Halmos 1965)

The product $X \times E$ is a linear lattice over reals relative to the partial ordering : $(x_1, e_1) \geqslant (x_2, e_2)$ iff $x_1 \geqslant x_2$, $e_1 \geqslant e_2$ and the operations :

(i) $(x_1, e_1) + (x_2, e_2) = (x_1 + x_2, e_1 + e_2)$;

(ii) $\eta(x, e) = (\eta x, \eta e)$ for every scalar η.

Let $\theta : X \times E \to F$ be a bilinear, order-preserving lattice homomorphism; the image of the ordered pair (x, e), $x \in X$, $e \in E$ under the map θ will be denoted by xe or ex. It is easy to see that $0e = x0 = 0$ and for every $x \in X$ and $e \in E$,

$$(-x)\, e = x(-e) = -(xe); \quad (-x)\,(-e) = xe$$

and $| xe | \leqslant | x | \; | e |$. Further, if $\{\alpha_n\}$ be a sequence in E with $0 - \lim_n \alpha_n \in E$ then for every $x \in X$,

$$0 - \lim_n (\alpha_n x) = (0 - \lim_n \alpha_n)\, x.$$

These are consequences of the order limit.

4. Simple Functions

Definition 4.1 — A function $f : S \to E$ is said to be simple if there exists a finite class of disjoint measurable sets B_i, $i = 1, 2, \ldots, n$ and finite number of non-zero elements c_i, $i = 1, 2, \ldots, n$ such that $f(x) = c_i$, $x \in B_i$, $i = 1, 2, \ldots n$ and $f(x) = 0$ if $x \notin \bigcup_{i=1}^{n} B_i$.

If f is simple and $\{c_1, c_2, \ldots, c_n\}$ is the set of distinct non-zero values of f, then $f(x) = \sum_{i=1}^{n} c_i \chi_{A_i}(x)$, where

$$A_i = \{x : f(x) = c_i\}, \; i = 1, 2, \ldots, n$$

is called the canonical representation of f.

We shall denote by $\mathcal{E}_E(Q)$ the class of all simple functions taking values in E. Henceforward, we assume X to be a conditionally complete linear lattice.

Definition 4.2 — For every measurable set B, we define $\varphi_B : \mathcal{E}_E(Q) \to F$ by the rule $\varphi_B(f) = \sum_{i=1}^{n} \iota(B \cap B_i)\, c_i$, where $f(x)$ has the canonical representation

$$f(x) = \sum_{i=1}^{n} c_i \chi_{B_i}(x), \;\; x \in S.$$

We shall write φ instead of φ_S.

For every measurable set B and $f \in \mathcal{E}_E(Q)$, the following are true :

(i) $\varphi_{\bigcup_{i=1}^{n} B_i}(f) = \sum_{i=1}^{n} \varphi_{B_i}(f)$, $B_i \cap B_j = \phi$, $i \neq j$;

(ii) $\varphi_B(f) \geqslant 0$ if $f \geqslant 0$ a.e. on B;

(iii) if $f (= c)$ be a constant function, then $\varphi_B(f) = C\iota(B)$;

(iv) if $f \geqslant 0$ and $B_1 \subseteq B_2$ then $\varphi_{B_1}(f) \leqslant \varphi_{B_2}(f)$.

Theorem 4.1 — For every measurable set B and $f, g \in \mathcal{E}_E(Q)$, we have

(i) $\varphi_B(\alpha f + \beta g) = \alpha\varphi_B(f) + \beta\varphi_B(g)$, α, β scalars;

(ii) $f \vee g$, $f \wedge g \in \mathcal{E}_E(Q)$;

(iii) if $f \geqslant g$ a.e. on B, then $\varphi_B(f) \geqslant \varphi_B(g)$;

(iv) $| \varphi_B(f) | \leqslant \varphi_B(| f |)$.

Proof is omitted.

Theorem 4.2 — Let $f_n \in \mathcal{E}_E(Q)$, $n = 1, 2, \ldots$ and $0 - \lim_n f_n = 0$. Then for every measurable set B,

$$0 - \lim_n \varphi_B(f_n(x)) = 0 = 0 - \lim_n \varphi_B(| f_n(x) |)$$

provided $\iota(B) < \infty$.

PROOF : By the definition of order limit (Nakano 1950), there exists a sequence $\omega_n \downarrow_0$ in E such that for $x \in S$; $| f_n(x) | \leqslant \omega_n$, $n = 1, 2, \ldots$.

It follows that

$$| \varphi_B(f_n(x)) | \leqslant \varphi_B(| f_n(x) |) \leqslant \varphi_B(\omega_n) = \omega_n \iota(B), \; n = 1, 2, \ldots .$$

As $\omega_n \iota(B) \downarrow_0$, we conclude that

$$0 - \lim_n \varphi_B(\, | f_n(x) | \,) = 0 = 0 - \lim_n \varphi_B(f_n(x)).$$

Corollary 4.1 — Let $f_n \in \mathcal{E}_E(Q)$, $n = 1, 2, \ldots$ and $0 - \lim_n f_n = 0$ a.e. on B; then $0 - \lim_n \varphi_B(f_n) = 0$.

Corollary 4.2 — Let $f_n, g_n \in \mathcal{E}_E(Q)$, $n = 1, 2, \ldots$ and

$$0 - \lim_n f_n = 0 - \lim_n g_n = f \text{ a.e. on } B.$$

Then

$$0 - \lim_n \varphi_B(f_n) = 0 - \lim_n \varphi_B(g_n).$$

PROOF : This follows from the corollary 4.1 and the Theorem 4.1.

Definition 4.3 — A sequence $\{f_n\}$ in $\mathcal{E}_E(Q)$ is said to be a Cauchy sequence iff $0 - \lim_{m,n} \varphi(\, | f_m(x) - f_n(x) | \,) = 0$ for every $x \in S$.

Theorem 4.3 — Let E be sequentially continuous where E^+ and X^+ are linearly ordered. If $\{f_n\}$ be a Cauchy sequence in $\mathcal{E}_E(Q)$ then there exists a subsequence $\{f_{n_k}\}$ which order-converges a.e. to a function $f : S \to E \cup \{\infty\}$.

PROOF : By hypothesis

$$0 - \lim_{m,n} \varphi(\, | f_m(x) - f_n(x) | \,) = 0. \qquad \ldots(4.1)$$

Let $0 < \epsilon \in E$ and set $S_{m,n}(\epsilon) = \{x : | f_m(x) - f_n(x) | > \epsilon\}$.

Then $\varphi(\, | f_m(x) - f_n(x) | \,) \geqslant \varphi_{S_{m,n}(\epsilon)}(\, | f_m(x) - f_n(x) | \,) > \epsilon\, \iota(S_{m,n}(\epsilon))$

so that we obtain

$$0 - \lim_{m,n} \iota(S_{m,n}(\epsilon)) = 0. \qquad \ldots(4.2)$$

Let $0 < u \in E$, $0 < v \in X$; choose $\epsilon = (v/2^k)$ where k is a positive integer. We construct for every k a sequence of positive integers : $N_1 = N(v/2)$, $N_2 = N(v/2^2), \ldots$ with the property that for every k and for every $m, n \geqslant N_k$,

$$\iota(S_{m,n}(u/2^k)) \leqslant \frac{v}{2^k}\cdot$$

Choose a sequence $n_1 = N_1$, $n_2 = \max\{(n_1 + 1), N_2\}, \ldots,$

$$n_k = \max\{(n_{k-1} + 1), N_k\}$$

so that $n_1 < n_2 < \ldots$ and $\{f_{n_k}\}$ is, indeed, a subsequence. Write

$$S_k = \left\{x : | f_{n_k}(x) - f_{n_{k+1}}(x) | > \frac{u}{2^k}\right\}$$

and $$F_k = \bigcup_{i=k}^{\infty} S_i.$$

If $x \notin F_k$ i.e., if $x \notin S_i$, $i \geqslant k$, then

$$| f_{n_i}(x) - f_{n_{i+1}}(x) | \leqslant \frac{u}{2^i} \quad \ldots(4.3)$$

Therefore, for $j > i \geqslant k$,

$$| f_{n_i}(x) - f_{n_j}(x) | \leqslant | f_{n_i}(x) - f_{n_{i+1}}(x) | + \ldots + | f_{n_{j-1}}(x) - f_{n_j}(x) |$$

$$\leqslant \frac{u}{2^i} + \ldots + \frac{u}{2^{j-1}}$$

$$< \frac{u}{2^{i-1}}.$$

As E is sequentially continuous, $(u/2^{i-1}) \downarrow 0$, and so we conclude that $0 - \lim_i f_{n_i}(x)$ exists in E, $x \notin S_i$, $i \geqslant k$.

We now define a function $f : S \to E \cup \{\infty\}$ by the rule: $f(x) = 0 - \lim_i f_{n_i}(x)$ if the limit exists in E, otherwise we put $f(x) = \infty$. Let A be the set of all points $x \in S$ for which $0 - \lim_n f_n(x)$ does not exist. Since for every k, $\{f_{n_k}\}$ is convergent in $S - F_k$, $A \subseteq F_k$ for every k. Therefore, $A \subseteq \bigcap_{k=1}^{\infty} F_k$ and

$$\iota(A) \leqslant \iota(F_k) \leqslant \frac{v}{2^{k-1}}$$

for every k. Consequently $\iota(A) = 0$. This proves the theorem.

5. Integration of Functions

Throughout the section we assume that F is sequentially continuous and E is operative and sequentially continuous. Let $(S, Q(S), \iota, X)$ be a measure space.

Definition 5.1 — A function $f : S \to E \cup \{\infty\}$ is said to be ι-integrable (or, integrable when there is no confusion) on $B \in Q(S)$ if there exists a Cauchy sequence $\{f_n\}$ in $\mathcal{E}_E(Q)$ 0-converging a.e. to f such that $0 - \lim_n \varphi_B(f_n) < \infty$.

In this case we write

$$\int_B f \, d\iota = 0 - \lim_n \varphi_B(f_n) \quad \ldots(5.1)$$

When $B = S$, we write $\int f \, d\iota$ for $\int_S f \, d\iota$.

In view of Corollary 4.2, $\int f d\iota$ is independent of the sequence $\{f_n\}$. In case f is a simple function, we take $f_n = f$ for every n and see that the Definitions 4.2 and 5.1 agree. It is immediate from the definition that if f is integrable and $f = g$ a.e. then g is also integrable.

Let $L_E(S, \iota)$ be the collection of all integrable functions $f : S \to E \cup \{\infty\}$. For brevity we shall write $L_E(S, \iota)$ by $L_E(\iota)$. Clearly $\mathcal{E}_E(Q) \subset L_E(\iota)$ which is a linear lattice over reals and the map $f \to \int f d\iota$ is linear.

Definition 5.2 — A sequence $\{f_n\}$ in $L_E(\iota)$ is said to be a Cauchy sequence iff

$$0 - \lim_{m,n} \int | f_m - f_n | d\iota = 0.$$

Definition 5.3 — A sequence $\{f_n\}$ in $L_E(\iota)$ is said to order converge to $f \in L_E(\iota)$ in the mean iff $0 - \lim_n \int | f_n - f | d\iota = 0$.

Theorem 5.1 — Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{E}_E(Q)$ 0-converging a.e. to $f \in L_E(\iota)$; then $\{f_n\}$ 0-converges to f in the mean.

PROOF : Since $\mathcal{E}_E(Q) \subset L_E(\iota)$,

$$0 - \lim_{n,m} \varphi(| f_n - f_m |) = 0 - \lim_{n,m} \int | f_n - f_m | d\iota = 0.$$

It is now easy to prove that for fixed m, $\{ | f_n - f_m | \}$ is a Cauchy sequence in $\mathcal{E}_E(Q)$ converging to $| f - f_m |$ a.e. Further, since $0 - \lim_n f_n = f$ a.e., there exists $A \in Q(S)$, $\iota(A) = 0$ and $\omega_n \downarrow_0$ in E such that whenever $x \notin A$,

$$| f_n(x) - f(x) | \leqslant \omega_n, \quad n = 1, 2, 3, \ldots;$$

consequently, for fixed m and $x \notin A$,

$$| \, | f_s(x) - f_m(x) | - | f(x) - f_m(x) | \, | \leqslant | f_s(\omega) - f(x) | \leqslant \omega_s, s = 1, 2, \ldots.$$

This shows that

$$0 - \lim_s | f_s - f_m | = | f - f_m | \text{ a.e.}$$

Therefore, $\int | f - f_m | d\iota = 0 - \lim_s \int | f_s - f_m | d\iota.$

This gives $0 - \lim_m \int | f - f_m | d\iota = 0$

Hence the theorem.

Theorem 5.2 — Let X^+ and E^+ be linearly ordered; if $\{f_n\} \in L_E(\iota)$ be a Cauchy sequene then there exists $f \in L_E(\iota)$ such that $0 - \lim_n \int | f_n - f | d\iota = 0$.

PROOF : By Theorem 5.1, for each n there exists a Cauchy sequence

$$g_n^r \in \mathcal{E}_E(Q), \quad r = 1, 2, \ldots$$

and a sequence $\omega_r \downarrow_0$ in F such that

$$\int | f_n - g_n^r | d\iota \leqslant \omega_r, \quad r = 1, 2, 3, \ldots. \qquad \ldots(5.2)$$

We have

$$\int | g_n^n - g_m^m | d\iota \leqslant \int | g_n^n - f_n | d\iota + \int | f_n - f_m | d\iota + \int | f_m - g_m^m | d\iota$$

$$\leqslant \omega_n + \int | f_n - f_m | d\iota + \omega_m;$$

accordingly

$$0 - \lim_{m,n} \int | g_n^n - g_m^m | d\iota \leqslant 0 - \lim_{m,n} \int | f_n - f_m | d\iota = 0.$$

Hence $\{g_n^n\}$ is a Cauchy sequence in $\mathcal{E}_E(Q)$ and admits of a subsequence $\{g_{n_k}^{n_k}\}$ (by Theorem 4.3) which 0-converges a.e. to a function $f : S \to E \cup \{\infty\}$.

Consequently, f is integrable and by Theorem 5.1,

$$0 - \lim_k \int | g_{n_k}^{n_k} - f | d\iota = 0.$$

The inequality

$$\int | g_n^n - f | d\iota \leqslant \int | g_n^n - g_{n_k}^{n_k} | d\iota + \int | g_{n_k}^{n_k} - f | d\iota \quad (n < n_k)$$

gives

$$0 - \lim_n \int | g_n^n - f | d\iota = 0. \qquad \ldots(5.3)$$

Also,

$$\int | f_n - f | d\iota \leqslant \int | f_n - g_n^n | d\iota + \int | g_n^n - f | d\iota$$

$$\leqslant \omega_n + \int | g_n^n - f | d\iota \quad \text{by (5.2).}$$

So, we obtain from (5.3)

$$0 - \lim_n \int | f_n - f | d\iota = 0.$$

This proves the theorem.

Theorem 5.3 — Let E^+ and F^+ be linearly ordered; if $\{f_n\}$ be a sequence in $L_E(\iota)$ 0-converging to $f \in L_E(\iota)$ in the mean, then there is a subsequence of $\{f_n\}$ which 0-converges to f a.e.

PROOF : Let $0 < \epsilon \in E$, $0 < \omega \in F$; for each positive integer k, there exists a positive integer $n(k)$ such that whenever $m \geqslant n(k)$,

$$\int | f_m - f | d\iota \leqslant \frac{\omega}{2^{2k}}.$$

We now choose a strictly increasing sequence $\{n_k\}$ as in Theorem 4.3. We shall show that $\{f_{n_k}\}$ is the desired subsequence.

As $n_k \geqslant n(k)$, we have $\int | f_{n_k} - f | \, d\iota \leqslant \frac{\omega}{2^{2k}}$; also the set

$$E_k = \left\{ x : | f_{n_k}(x) - f(x) | > \frac{\epsilon}{2^k} \right\}$$

is measurable. Accordingly, we have

$$\frac{\omega}{2^{2k}} \geqslant \int | f_{n_k} - f | \, d\iota \geqslant \int_{E_k} | f_{n_k} - f | \, d\iota > \frac{\epsilon}{2^k} \iota(E_k)$$

$$\text{i.e., } \frac{\omega}{2^k} > \epsilon\iota(E_k). \qquad \ldots(5.4)$$

Let $F_k = \bigcup_{i=k}^{\infty} E_i$, so that F_k is measurable and for $x \in S - F_k$, that is, for

$$x \notin E_i, \; i \geqslant k$$

$$| f_{n_k} - f | \leqslant \frac{\epsilon}{2^k} \cdot$$

This shows that $\{f_{n_i}\}$ 0-converges to f on $S - F_k$.

The set $D = \bigcap_{k=1}^{\infty} F_k$ is measurable and

$$\epsilon\iota(D) \leqslant \epsilon\iota(F_k) \leqslant \epsilon \sum_{i=k}^{\infty} \iota(E_i) < \sum_{i=k}^{\infty} \frac{\omega}{2^i} = \frac{\omega}{2^k}$$

for every k by (5.4). Therefore, $\epsilon\iota(D) = 0$, and hence $\iota(D) = 0$. Since

$$S - D = \bigcup_{k=1}^{\infty} (S - F_k),$$

we deduce that on $S - D$ the sequence $\{f_{n_k}\}$ 0-converges to f. The set A on which $\{f_{n_k}\}$ does not converge is a subset of D. Hence $\{f_{n_k}\}$ 0-converges a.e. to f. This proves the theorem.

Theorem 5.4 — Let X^+, E^+ and F^+ be linearly ordered; if $\{f_n\}$ be a Cauchy sequence in $L_E(\iota)$ 0-converging almost everywhere to a function $f : S \rightarrow E$, then f is integrable and $\{f_n\}$ 0-converges to f in the mean; moreover $\int f d\iota = 0 - \lim_n \int f_n d\iota$.

PROOF: By Theorem 5.2, there exists $g \in L_E(\iota)$ to which $\{f_n\}$ converges in the mean; accordingly

$$0 - \lim_n \int | f_n - g | \, d\iota = 0.$$

By Theorem 5.3, we deduce that there exists a subsequence $\{f_{n_k}\}$ 0-converging a.e. to g. $\{f_{n_k}\}$ also 0-converges a.e. to f. It follows that $g = f$ a.e. Hence f is integrable, and $0 - \lim_n \int f_n d\iota = \int g d\iota = \int f d\iota$. Hence the theorem.

Theorem 5.5 — Let E^+, F^+ and X^+ be linearly ordered; if $\{f_n\}$ be a non-decreasing sequence of functions in $L_E(\iota)$ and if $\sup_n \int f_n d\iota$ exists in F, then the function $\sup_n f_n$ is integrable and

$$\int (\sup_n f_n)\, d\iota = \sup_n \int f_n d\iota = 0 - \lim_n \int f_n d\iota.$$

PROOF: Since $(\int f_n d\iota) \uparrow_n$ we have $0 - \lim_n \int f_n d\iota = \sup_n \int f_n d\iota$. Consequently, $\{f_n\}$ is a Cauchy sequence in $L_E(\iota)$, and by Theorem 5.2, $\{f_n\}$ converges in the mean to a function $g \in L_E(\iota)$. It follows from Theorem 5.3 that there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ 0-converging a.e. to g. But $\{f_{n_k}\}$ 0-converges to $f = \sup_n f_n$ as well. Therefore, it follows that $g(x) = f(x)$ a.e. Hence f is integrable, and $\{f_n\}$ converges in the mean to f. So,

$$\int f d\iota = 0 - \lim_n \int f_n d\iota = \sup_n \int f_n d\iota.$$

This proves the theorem.

Corollary 5.1 — Let $\{f_n\}$ be a sequence in $L_E(\iota)$ and g be a positive integrable function such that $| f_n | \leqslant g$ for every n. Then the function $\sup_n f_n$ is integrable and

$$\sup_n \int f_n d\iota \leqslant \int (\sup_n f_n)\, d\iota.$$

PROOF: For each n, we set

$$g_n = \sup_{1 \leqslant i \leqslant n} f_i.$$

So, $\{g_n\}$ becomes a non-decreasing sequence of functions in $L_E(\iota)$ such that

$$\sup_n g_n = \sup_n f_n.$$

Since $g_n \leqslant g$ for every n, we have

$$\int g_n d\iota \leqslant \int g d\iota \quad \text{for every } n.$$

Accordingly, $\sup_n \int g_n d\iota$ exists in F, and by Theorem 5.5, we conclude that

$$\sup_n f_n = \sup_n g_n$$

is integrable. It now follows easily that

$$\sup_n \int f_n d\iota \leqslant \int (\sup_n f_n)\, d\iota.$$

Corollary 5.2 — Let $\{f_n\}$ be a non-increasing sequence in $L_E(\iota)$. If $\inf_n \int f_n d\iota$ exists in F then $\inf_n f_n$ is integrable and

$$\int (\inf_n f_n)\, d\iota = \inf_n \int f_n d\iota = 0 - \lim_n \int f_n d\iota$$

Theorem 5.6 — Let A be a countable directed set; f_α, $\alpha \in A$ be a family of integrable functions and also g be an integrable function. If $| f_\alpha | \leqslant g$ for every $\alpha \in A$ then the function $0 - \overline{\lim}_\alpha f_\alpha$ is integrable and

$$0 - \overline{\lim}_\alpha \int f_\alpha d\iota \leqslant \int (0 - \overline{\lim}_\alpha f_\alpha)\, d\iota.$$

PROOF : Let the elements of A be enumerated as

$$\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$$

Choose $\alpha \in A$ and write for a fixed n,

$$p_n = \alpha_n,\ p_{n+1} = \max\{\alpha, p_n\}, \ldots, p_{n+s} = \max\{\alpha, p_{n+s-1}\}, \ldots$$

so that $\quad p_n \leqslant p_{n+1} \leqslant \ldots \leqslant p_{n+s} \leqslant \ldots$.

Write $g_n = \sup\limits_{s \geqslant 0} f_{p_{n+s}}$; so $g_n \in L_E(\iota)$ for every n by Corollary 5.1 and

$$\sup_s \int f_{p_{n+s}} d\iota \leqslant \int g_n d\iota \qquad \ldots(5.5)$$

Clearly, the sequence $\{g_n\}$ is non-increasing and $g_n \geqslant f_{p_n} = f_{\alpha_n} \geqslant -g$ for every n; accordingly, since F is conditionally σ-complete, $\inf_n \int g_n d\iota$ exists and so by Corollary 5.2

$$\inf_n g_n = \inf_n (\sup_{s \geqslant 0} f_{p_{n+s}}) = 0 - \overline{\lim}_\alpha f_\alpha$$

is integrable. Also

$$\int \inf_n g_n d\iota = \inf_n \int g_n d\iota \geqslant \inf_n (\sup_{s \geqslant 0} \int f_{p_{n+s}} d\iota),$$

i.e. $$\int (0 - \overline{\lim}_\alpha f_\alpha)\, d\iota \geqslant 0 - \overline{\lim}_\alpha \int f_\alpha d\iota.$$

This proves the theorem.

Theorem 5.7 — Let E^+, F^+ and X^+ be linearly ordered; suppose that a sequence $\{f_n\}$ in $L_E(\iota)$ 0-converges to $f : S \to E$ a.e. If there exists an integrable function g such that $| f_n | \leqslant g$ a.e. for every n then f is integrable and

$$\int f d\iota = 0 - \lim_n \int f_n d\iota.$$

PROOF : By the given condition

$$0 - \lim_{m,n} | f_m - f_n | = 0 \quad \text{a.e.}$$

and as such $0 - \overline{\lim}_{m,n} | f_m - f_n | = 0$ a.e.

Let N denote the set of natural numbers. The set $N \times N$ is countable and is directed by the order $(m, n) \leqslant (m', n')$ iff $m \leqslant m'$, $n \leqslant n'$.

By Theorem 5.6 we obtain $0 - \overline{\lim}_{m,n} \int | f_n - f_m | \, d\iota = 0$.

Therefore, $\{f_n\}$ is a Cauchy sequence in $L_E(\iota)$ 0-converging to f a.e.; thus by Theorem 5.4, f is integrable and $\int f d\iota = 0 - \lim_n \int f_n d\iota$.

This proves the theorem.

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