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ON A THEOREM OF BRIAN FISHER

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A fixed point theorem for a self-mapping of a metric space which is not necessarily continuous has been presented. Fixed point theorem for a pair of continuous self-mappings on a bounded metric space has also been presented.

Let (X, d) denote a metric space. A mapping $T : X \to X$ is said to be a contraction mapping if there exists a real constant λ, $0 \leqslant \lambda < 1$, such that

$$d(Tx, Ty) \leqslant \lambda d(x, y) \quad \text{for all } x, y \in X. \qquad \ldots(1)$$

Banach proved that a contraction mapping on a complete metric space has a unique fixed point. But contraction mappings are necessarily continuous. However, in recent years multitude of fixed point theorems for different types of mappings which are not necessarily continuous have been obtained (see Rhoades 1977).

Recently Gupta and Ranganathan (1975) obtained the following theorem for a mapping $T : X \to X$ which is not necessarily continuous for $p = 0$.

Theorem 1 — Let T be a self-mapping of a complete metric space (X, d) such that

$$d(T^{p+1}x, T^{p+2}y) \leqslant \alpha d(T^p x, T^{p+1}x) + \beta d(T^{p+1}y, T^{p+2}y) + \gamma d(T^p x, T^{p+1}y) \qquad \ldots(2)$$

where $x, y \in X$, p, a non-negative integer and α, β, γ are constants such that $\alpha, \beta, \gamma \geqslant 0$ with $\alpha + \beta + \gamma < 1$.

Then T has a unique fixed point.

More recently Fisher (1978) obtained the following:

Theorem 2 — Let S and T be two self-mappings of a complete metric space (X, d) satisfying the inequality:

$$[d(Sx, Ty)]^2 \leqslant bd(x, Sx)\, d(y, Ty) + cd(x, Ty)\, d(y, Sx) \qquad \ldots(3)$$

for all $x, y \in X$, where $0 \leqslant b < 1$, $c \geqslant 0$; then S and T have a common fixed point.

Further, if $0 \leqslant b, c < 1$, then each of S and T has a unique fixed point and these two fixed points coincide.

Now for $S = T$ the inequality (3) reduces to

$$[d(Tx, Ty)]^2 \leqslant bd(x, Tx)\, d(y, Ty) + cd(x, Ty)\, d(y, Tx). \qquad \ldots(4)$$

Then main purpose of this note is to present a fixed point theorem of the type of Theorem 1 for a self-mapping T of a complete metric space (X, d) where T satisfies

$$[d(T^{p+1}x, T^{p+2}y)]^2 \leqslant bd(T^px, T^{p+1}x)\, d(T^{p+1}y, T^{p+2}y) + cd(T^px, T^{p+2}y)\, d(T^{p+1}y, T^{p+1}x) \qquad \ldots(5)$$

where $x, y \in X$, p, a non-negative integer and b, c are constants such that

$$0 \leqslant b < 1, \quad c \geqslant 0.$$

A fixed point theorem for a pair of mappings is also presented. First we present the following :

Theorem 3 — Let (X, d) be a complete metric space and let $T: X \to X$ satisfy (5). Then T has a fixed point if either $p = 0$ or T is continuous. Further, if $0 \leqslant b, c < 1$, then the fixed point of T is unique.

PROOF : We prove the theorem for $p = 0$. The proof in the general case follows on similar lines.

Now, for $p = 0$ we get from (5)

$$[d(Tx, T^2y)]^2 \leqslant bd(x, Tx)\, d(Ty, T^2y) + cd(x, T^2y)\, d(Tx, Ty).$$

For an arbitrary point $x_0 \in X$ we define $\{x_n\}$ as

$$x_n = T^n x_0, \quad x_{n+1} = Tx_n, \quad n = 0, 1, 2, \ldots .$$

Then

$$[d(x_1, x_2)]^2 = [d(Tx_0, T^2x_0)]^2$$
$$\leqslant bd(x_0, Tx_0)\, d(Tx_0, T^2x_0) + cd(x_0, T^2x_0)\, d(Tx_0, Tx_0)$$

i.e. $\quad d(x_1, x_2) \leqslant bd(x_0, x_1)$.

Similarly

$$[d(x_2, x_3)]^2 = [d(Tx_1, T^2x_1)]^2$$
$$\leqslant bd(x_1, Tx_1)\, d(Tx_1, T^2x_1) + cd(x_1, T^2x_1)\, d(Tx_1, Tx_1)$$

i.e. $\quad d(x_2, x_3) \leqslant bd(x_1, x_2) \leqslant b^2 d(x_0, x_1)$

and, in general, $d(x_n, x_{n+1}) \leqslant b^n d(x_0, x_1)$.

Since $b < 1$, $\{x_n\}$ is a Cauchy sequence.

Now, from the completeness of X, it follows that $\lim_{n \to \infty} x_n = \xi \in X$.

We then, have

$$[d(T\xi, T^n x_0)]^2 \leqslant bd(\xi, T\xi)\, d(Tx_{n-2}, T^2 x_{n-2}) + cd(\xi, T^n x_0)\, d(T\xi, Tx_{n-2})$$

and on letting $n \to \infty$ we get $[d(T\xi, \xi)]^2 = 0$, which implies that ξ is a fixed point of T.

Now suppose that $0 \leqslant b,\ c < 1$ and η, $\eta \neq \xi$, is a second fixed point of T.

Then

$$\begin{aligned}[d(\xi, \eta)]^2 &= [d(T\xi, T^2\eta)]^2 \\ &\leqslant bd(\xi, T\xi)\, d(T\eta, T^2\eta) + cd(\xi, T^2\eta)\, d(T\xi, T\eta).\end{aligned}$$

Since $c < 1$, it follows that $\xi = \eta$ and so the fixed point of T is unique.

Finally, we note that if $p > 0$, $\{x_n\}$ is still a Cauchy-sequence and the existence of the fixed point follows from the continuity of T. This completes the proof of Theorem 3.

We illustrate Theorem 3 by the following example:

Example 1 — Let $X = [0, 1]$ with the usual metric and let $T : X \to X$ be defined as

$$\begin{aligned} Tx &= 0 \quad \text{for} \quad x \neq \tfrac{1}{3} \\ &= 1 \quad \text{for} \quad x = \tfrac{1}{3}. \end{aligned}$$

Taking $b = 0$, $c = 2$, $y = 0$ and $x = \frac{1}{3}$ we see that T does not satisfy (4), but it satisfies (5) with $p = 1$. Lastly we present the following:

Theorem 4 — Let S and T be two mappings of a bounded complete metric space (X, d) into itself such that T is continuous and S and T satisfy

$$[d(Sx, T^2y)]^2 \leqslant \lambda \max\{d(x, Sx)\, d(Ty, T^2y),\ d(x, T^2y)\, d(Sx, Ty)\} \quad \ldots(6)$$

$$\forall\ x, y \in X, \text{ where } 0 \leqslant \lambda < 1.$$

Further, suppose that $ST = TS$. Then T has a unique fixed point ξ and ξ is the unique common fixed point of S and T.

PROOF : Since X is bounded, we put $M = \text{Sup}\{d(x, y) : x, y \in X\} < \infty$. Let $x_0 \in X$ be arbitrary. Then

$$\begin{aligned}&[d((ST)^n x_0, T(ST)^n x_0)]^2 \\ &\quad \leqslant \lambda \max\{d(S^{n-1}T^n x_0, S^n T^n x_0)\, d(T^n S^n x_0, T^{n+1} S^n x_0), \\ &\qquad d(S^{n-1}T^n x_0, T^{n+1} S^n x_0)\, d(T^n S^n x_0, S^n T^n x_0)\}\end{aligned}$$

i.e. $$[d((ST)^n x_0, T(ST)^n x_0)] \leqslant \lambda d(S^{n-1}T^n x_0, (ST)^n x_0)$$
$$\cdots \quad \cdots \quad \cdots$$
$$\cdots \quad \cdots \quad \cdots$$
$$\leqslant \lambda^n M.$$

Similarly, $d(T(ST)^n x_0, (ST)^{n+1} x_0) \leqslant \lambda^n M, \ n = 1, 2, \ldots .$

Since $\lambda < 1$ it follows that the sequence:

$$\{x_0, Tx_0, (ST)\, x_0, \ldots, (ST)^n x_0, T(ST)^n x_0, \ldots\}$$

is a Cauchy sequence in the complete metric space (X, d) with a limit point ξ in X.

Now, from the continuity of T we have $T\xi = \lim_{n\to\infty} T(ST)^n x_0 = \lim_{n\to\infty} (ST)^n x_0 = \xi$ and then $\lim_{n\to\infty} T^2(ST)^n x_0 = \xi$.

Further,

$$[d(S\xi, T^2(ST)^n x_0)]^2$$
$$\leqslant \lambda \max \{d(\xi, S\xi)\, d(T(ST)^n x_0, T^2(ST)^n x_0), d(\xi, T^2(ST)^n x_0) \times d(S\xi, T(ST)^n x_0)\}$$

Letting $n \to \infty$, we get $[d(S\xi, \xi)]^2 = 0$. It follows that $S\xi = \xi$ and so ξ is a common fixed point of S and T. Unicity of ξ is obvious.

References

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