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SYSTEM OF GRIFFITH CRACKS LYING AT THE INTERFACE OF TWO BONDED DISSIMILAR ELASTIC HALF-PLANES

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The stress field in the vicinity of a system of m Griffith cracks located at the interface of two bonded dissimilar elastic half-planes is determined. Following a procedure of Lowengrub and Sneddon, the problem has been reduced to a system of simultaneous integral equations which are equivalent to Riemann boundary value problem with closed form solution. The problem, when number of cracks is three, is treated in detail.

INTRODUCTION

Lowengrub and Sneddon (1973) have considered the problem of determining the stress field due to the presence of a Griffith crack located at the interface of two bonded dissimilar elastic half-planes. They assume that the deformation of the composite solid is due to the application of a prescribed pressure to the upper and lower surfaces of the crack. The representation of the displacement in terms of Fourier transform reduces the problem to that of solving a set of simultaneous dual integral equations which are shown to be equivalent to Riemann boundary value problem which has a closed form solution. Lowengrub (1975) used this technique to determine the stress field due to the presence of two coplanar cracks located at the interface of two bonded dissimilar half-planes. In this case the problem reduces to the solution of a system of simultaneous triple integral equations. Lowengrub mentions in his paper that the technique can be extended to any number of equally spaced cracks. However, from his paper, it appears that the number of cracks should be even in number. He has not given the details of the general problem and has not shown how the particular cases of single and double cracks may be deduced.

In view of the growing importance of the problems of dissimilar elastic solids containing cracks at the interface, it will be useful to give details of the problem of m cracks. Using Fourier transform the problem reduces to the following set of simultaneous equations:

$$F_c\,[\alpha\varphi(\xi) + \beta\psi(\xi);\, x] = f_1(x),\; x \in L_1 \qquad \ldots(1.1)$$

$$F_s\,[\beta\varphi(\xi) + \alpha\psi(\xi);\, x] = f_2(x),\; x \in L_1 \qquad \ldots(1.2)$$

$$F_c\,[\varphi(\xi);\, x] = 0,\; x \in L_1' \qquad \ldots(1.3)$$

$$F_s\,[\psi(\xi);\, x] = 0,\; x \in L_1'\,. \qquad \ldots(1.4)$$

Here, F_c, F_s represent Fourier cosine and sine transforms respectively;

$$L_1 = U_{j=1}^{n} (a_j, b_j), L_1' = R' - L_1,$$

R' is the positive real axis; $0 \leqslant a_1 < b_1 < a_2 < b_2 \, .. \, a_n < b_n$.

We shall begin by solving this set of integral equations in section 2. This system of integral equations shall be reduced to Riemann boundary value problem which is known to have closed form solution. This solution will be used to study the distribution of stress due to the presence of m cracks located at the interface of two bonded dissimilar half-planes. It is assumed that the cracks are symmetrically situated with respect to y-axis taken perpendicular to the interface. The cracks are defined by the relation $y = 0, a_j \leqslant | x | \leqslant b_j$, $j = 1, 2, 3 \ldots n$. The number of cracks shall be odd or even according as $a_1 = 0$ or $a_1 \neq 0$. The cases of single or double cracks, which have been treated by Lowengrub and Sneddon (1973) and Lowengrub (1975), can be derived from the general case treated here by taking $n = 1$, $a_1 = 0$ and $n = 1$, $a_1 \neq 0$ respectively. These results are given in section 4. Lowengrub (1975) has mentioned that the constant d_2' has to be calculated numerically. It is interesting to note that this constant can be obtained in closed form. The section 5 contains a detailed study of the case when the number of cracks at the interface is three. When a constant pressure p_0 is applied at the faces of the cracks, the normal stress component across the interface is given by

$$p_0 [1 - \{(x^2 - a^2)(x^2 - b^2)(x^2 - c^2)\}^{-1/2} (x^3 + d_2' x) \cos \omega\theta$$
$$+ 2 \{(a + b - c) x^2 - d_3'')\} \sin \omega\theta], \quad x > c,$$

where $\theta = \log \{(x + a)(x - b)(x + c)/(x - a)(x + b)(x - c)\}$. Here d_2', d_3'' are arbitrary real constants which can be expressed in closed form in terms of generalized hypergeometric function

$$F_D [a_1; b_1, b_2, \ldots, b_n; c; x_1, x_2, \ldots, x_n].$$

We assume that stress components satisfy the conditions

$$\sigma_{yy}(x, 0) = O(x^{-1}), \quad \sigma_{xy}(x, 0) = O(x^{-1}) \quad \text{as} \quad x \to \infty.$$

2. Solution of the Integral Equations

We begin by solving the system of integral eqns. (1.1) – (1.4). Let

$$\left. \begin{aligned} F_c [\varphi(\xi); x] &= r_1(x), \, x \in L_1 \\ &= 0 \quad , x \in L_1' \end{aligned} \right\} \qquad \ldots(2.1)$$

$$\left. \begin{aligned} F_s [\psi(\xi); x] &= s_1(x), \, x \in L_1 \\ &= 0 \quad , x \in L_1'. \end{aligned} \right\} \qquad \ldots(2.2)$$

As in Lowengrub and Sneddon (1973), it can be easily shown that

$$F_s\,[\varphi(\xi);\, x] = \frac{1}{\pi}\int_L \frac{r(u)}{x-u}\,du \qquad \text{...(2.3)}$$

$$F_c\,[\psi(\xi);\, x] = \frac{1}{\pi}\int_L \frac{s(u)}{u-x}\,du \qquad \text{...(2.4)}$$

where $r(u)$ and $s(u)$ are even and odd extensions of $r_1(u)$, $s_1(u)$ respectively to the interval L_2, where

$$L_2 = U_{j=1}^{n}\,(-\,b_j,\,-\,a_j),\ L = L_1 \cup L_2.$$

With the help of these equations the system of integral eqns. (1.1) – (1.4) reduce to the following singular integral equations

$$\alpha r(x) + (\beta/\pi)\int_L \frac{s(u)}{u-x}\,du = \hat{f}_1(x),\ x \in L \qquad \text{...(2.5)}$$

$$\alpha s(x) - (\beta/\pi)\int_L \frac{r(u)}{u-x}\,du = \hat{f}_2(x),\ x \in L \qquad \text{...(2.6)}$$

where $\hat{f}_1(x)$ and $\hat{f}_2(x)$ are respectively the even and odd extensions of $f_1(x)$ and $f_2(x)$ to L_2. If we write

$$\lambda(x) = s(x) - ir(x) \qquad \text{...(2.7)}$$

then the above two equations can be written as a single equation

$$i\alpha\lambda(x) + (\beta/\pi)\int_L \frac{\lambda(u)}{u-x}\,du = f(x),\ x \in L \qquad \text{...(2.8)}$$

where $f(x) = \hat{f}_1(x) + i\hat{f}(x)$. If we write

$$\Lambda(z) = \frac{1}{2\pi i}\int_L \frac{\lambda(u)}{u-z}\,du \qquad \text{...(2.9)}$$

and on using Pelmelj formula (Muskhelishvili 1963)

$$\Lambda^+(x) - \Lambda^-(x) = \lambda(x),\ \Lambda^+(x) + \Lambda^-(x) = \frac{1}{\pi i}\int_L \frac{\lambda(u)}{u-x}\,du \qquad \text{...(2.10)}$$

the integral eqn. (2.8) reduces to the following Riemann boundary value problem

$$\Lambda^+(x) = -\,k\Lambda^-(x) - i(\alpha+\beta)^{-1} f(x),\ x \in L \qquad \text{...(2.11)}$$

where $k = (\beta - \alpha)/(\beta + \alpha)$. The solution of this problem, as given in Muskhelishvili (1963), is

$$\Lambda(z) = \frac{X(z)}{2\pi(\beta + \alpha)} \int_L \frac{f(t)\, dt}{X^+(t)\,(t - z)} + P(z)\, X(z) \qquad \ldots(2.12)$$

where $P(z) = d_1 z^{n-1} + d_2 z^{n-2} + \ldots + d_n, d_1, d_2, \ldots, d_n$ are arbitrary complex constants and $X(z)$ is the solution of homogeneous Riemann boundary value problem

$$X^+(t) = -\, k X^-(t),\ t \in L.$$

The solution of this problem is (Muskhelishvili 1963)

$$X(z) = \prod_{j=1}^{n} [(z - a_j)(z + b_j)]^{-(1-2i\omega)/2} [(z + a_j)(z - b_j)]^{-(1+2i\omega)/2},\ a_1 \neq 0$$

$$= (z + b_1)^{-(1-2i\omega)/2} (z - b_1)^{-(1+2i\omega)/2} \prod_{j=2}^{n} [(z - a_j)(z + b_j)]^{-(1-2i\omega)/2}$$

$$\times\ [(z + a_j)(z - b_j)]^{-(1+2i\omega)/2},\quad a_1 = 0 \qquad \ldots(2.13)$$

where $\omega = (2\pi)^{-1} \log k$. In case $f(x)$ is a polynomial

$$\int_L \frac{f(t)\, dt}{X^+(t)\,(t - z)} = \frac{2\pi i}{k + 1} [\{f(z)/X(z)\} - L(z)] \qquad \ldots(2.14)$$

where

$$L(z) = \lim_{R \to \infty} \frac{1}{2\pi} \int_0^{2\pi} \frac{f(Re^{i\theta})\, Re^{i\theta}}{X(Re^{i\theta})\,(Re^{i\theta} - z)}\, d\theta. \qquad \ldots(2.15)$$

Hence from (2.12) we have

$$\Lambda(z) = i(2\beta)^{-1} [f(z) - L(z)\, X(z) + P(z)\, X(z)]. \qquad \ldots(2.16)$$

3. A System of Griffith Cracks at the Interface of two Bonded Dissimilar Elastic Half-planes

In this section, we shall apply the solution of the last section for studying the distribution of stresses in the vicinity of m Griffith cracks located at the interface of two half-planes. The cracks are located symmetrically with respect to y-axis which is taken perpendicular to the interface. The two half-planes $y > 0$ and $y < 0$ are occupied by elastic materials having elastic constants μ_1, k_1 and μ_2, k_2 respectively. Here μ_i denote the modulus of rigidity and $k_i = 3 - 4\eta_i$, $i = 1, 2$, where η_i is Poisson's ratio. The cracks location is given by $y = 0$, $a_j \leqslant |\, x \,| \leqslant b_j$, $j = 1, 2, 3, \ldots, n$. The lower and upper surfaces of the cracks are subjected to a prescribed pressure $p(x)$. Inside the cracks we have the conditions

$$\sigma_{yy}(x, 0) = \sigma_{yy}(x, 0-) = -p(x), \quad x \in L \qquad \text{...(3.1)}$$

$$\sigma_{xy}(x, 0+) = \sigma_{xy}(x, 0-) = 0, \quad x \in L. \qquad \text{...(3.2)}$$

On the region outside the cracks at the interface we have the continuity conditions

$$U(x, 0+) = U(x, 0-), \;\; V(x, 0+) = V(x, 0-), \quad x \in L' \qquad \text{...(3.3)}$$

$$\sigma_{yy}(x, 0+) = \sigma_{yy}(x, 0-), \;\; \sigma_{xy}(x, 0+) = \sigma_{xy}(x, 0-), \quad x \in L'. \qquad \text{...(3.4)}$$

In order to simplify the calculation, we suppose that $p(x)$ is an even function of x. The solution of displacement equations can be written as

$$U(x, y) = \begin{cases} F_s\,[\xi^{-1}\{A_1 - k_1^{-1}(A_1 - B_1)\,\xi y\}\,e^{-\xi y};\, \xi \to x],\; y > 0 \\ F_s\,[\xi^{-1}\{A_2 + k_2^{-1}(A_2 + B_2)\,\xi y\}\,e^{\xi y},\;\; \xi \to x],\; y < 0 \end{cases} \qquad \text{...(3.5)}$$

$$V(x, y) = \begin{cases} F_c\,[\xi^{-1}\{B_1 - k_1^{-1}(A_1 - B_1)\,\xi y\}\,e^{-\xi y};\, \xi \to x],\; y > 0 \\ F_c\,[\xi^{-1}\{B_2 - k_2^{-1}(A_2 + B_2)\,\xi y\}\,e^{\xi y};\;\; \xi \to x],\; y < 0. \end{cases} \qquad \text{...(3.6)}$$

Following Lowengrub and Sneddon (1973), it can be shown that the conditions (3.1) – (3.4) lead to the system of integral eqns. (1.1) – (1.4), where

$$k_1(k_2 + \Gamma)\,(1 + k_1\Gamma)\,A_1 = [k_1\Gamma + \tfrac{1}{2}(k_1k_2 + 1)]\,\varphi(\xi) - \tfrac{1}{2}(k_1k_2 - 1)\,\psi(\xi)$$

$$k_1(k_2 + \Gamma)\,(1 + k_1\Gamma)\,B_1 = -\tfrac{1}{2}(k_1k_2 - 1)\,\varphi(\xi) + [k_1\Gamma + \tfrac{1}{2}(k_1k_2 + 1)]\,\psi(\xi)$$

$$\Gamma = \frac{\mu_1}{\mu_2},\; \alpha = (k_1 - 1)\,\Gamma - (k_2 - 1),\; \beta = (k_1 + 1)\,\Gamma + (k_2 + 1),$$

$$f(x) = \frac{k_1(k_2 + \Gamma)\,(1 + k_1\Gamma)\,p(x)}{\mu_1}. \qquad \text{...(3.7)}$$

With the help of (2.1) – (2.4) it can be easily shown that

$$\sigma_{yy}(x, 0+) = \sigma_{yy}(x, 0-) = -\frac{\beta\mu_1}{\pi k_1(k_2 + \Gamma)\,(1 + k_1\Gamma)} \times \int_L \frac{s(u)}{u - x}\,du, \quad x \in L_1' \qquad \text{...(3.8)}$$

$$\sigma_{xy}(x, 0+) = \sigma_{xy}(x, 0-) = \frac{\beta\mu_1}{k_1(k_2 + \Gamma)\,(1 + k_1\Gamma)\,\pi} \times \int_L \frac{r(u)}{u - x}\,du, \quad x \in L_1' \qquad \text{...(3.9)}$$

Also, as in Lowengrub (1975), the continuity conditions (3.3) are satisfied if

$$F_s\,[\xi^{-1}\varphi(\xi);\,x] = 0,\;\; F_c\,[\xi^{-1}\psi(\xi);\,x] = 0,\quad x \in L_1'.$$

Substituting the values of $\varphi(\xi)$ and $\psi(\xi)$ from (2.1) and (2.2) in the above equations and after interchanging the order of integration we get

$$\int_{a_j}^{b_j} s_1(u)\,du = 0,\; \int_{a_j}^{b_j} r_1(u)\,du = 0,\; j = 1, 2, 3, \ldots, n$$

Hence from (2.7) we have

$$\int_{a_j}^{b_j} \lambda(u)\,du = 0,\; j = 1, 2, \ldots, n. \qquad \ldots(3.10)$$

The constants $d_1, d_2, \ldots, d_n$ of the polynomial $P(z)$ in (2.12) can be determined with the help of the conditions (3.10).

4. Particular Cases

The problems of single and double cracks has been studied by Lowengrub and Sneddon (1973) and Lowengrub (1975). These results can be obtained as particular cases of the general problem studied in the previous section.

(a) *Single Crack at the Interface Opened by Constant Pressure*

In this case the crack is defined by $y = 0,\; -1 \leqslant x \leqslant 1$ and L_1 is the interval (0, 1). If the crack is opened by constant pressure $p(x) = p_0$, then from (1.1) – (1.4) we have

$$F_c\,[\alpha\varphi(\xi) + \beta\psi(\xi);\,x] = f_0,\;\; 0 \leqslant x \leqslant 1 \qquad \ldots(4.1)$$

$$F_s\,[\beta\phi(\xi) + \alpha\psi(\xi);\,x] = 0,\;\; 0 \leqslant x \leqslant 1 \qquad \ldots(4.2)$$

$$F_c\,[\varphi(\xi);\,x] = 0,\;\; x > 1 \qquad \ldots(4.3)$$

$$F_s\,[\psi(\xi);\,x] = 0\;\; x > 1. \qquad \ldots(4.4)$$

From the results derived in section 2, we have

$$\left.\begin{aligned} &X(z) = (z+1)^{-(1-2i\omega)/2}\,(z-1)^{-(1+2i\omega)/2},\; P(z) = d_1 \\ &L(z) = f_0(z - 2i\omega),\; \Lambda(z) = if_0(2\beta)^{-1}\,[1 - (z + d_1)\,X(z)] \\ &f_0 = k_1(k_2 + \Gamma)\,(1 + k_1\Gamma)\,p_0/\mu_1. \end{aligned}\right\} \qquad \ldots(4.5)$$

Hence

$$r(x) = -f_0(\beta^2 - \alpha^2)^{-1/2}\,(1 - x^2)^{-1/2}\,(x \sin\omega\theta - \omega\cos\omega\theta),\;\; 0 \leqslant x \leqslant 1 \qquad \ldots(4.6)$$

$$s(x) = -f_0(\beta^2 - \alpha^2)^{-1/2}\,(1 - x^2)^{-1/2}\,(x \cos\omega\theta + \omega\sin\omega\theta),\;\; 0 \leqslant x \leqslant 1 \qquad \ldots(4.7)$$

and for $x > 1$ the stress components are

$$\sigma_{yy}(x, 0+) = p_0\,[(x^2 - 1)^{-1/2}\,(x \cos \omega\theta + \omega \sin \omega\theta) - 1] \qquad ...(4.8)$$

$$\sigma_{xy}(x, 0+) = p_0\,(x^2 - 1)^{-1/2}\,(x \sin \omega\theta - \omega \cos \omega\theta) \qquad ...(4.9)$$

where $\theta = \log \{(x + 1)/(x - 1)\}$.

(b) *Two Collinear Griffith Cracks at the Interface*

In the case of two collinear Griffith cracks at the interface, $L_1 = (a, b)$ and the cracks are defined by $y = 0,\ a \leqslant |x| \leqslant b,\ a \neq 0$. If the cracks are opened by applying constant pressure p_0 at the inner faces of the cracks, then the system of integral eqns. (1.1) to (1.4) reduces to the following set of simultaneous triple integral equations

$$F_c\,[\varphi(\xi); x] = 0,\ 0 < |x| < a,\ |x| > b \qquad ...(4.10)$$

$$F_s\,[\psi(\xi); x] = 0,\ 0 < |x| < a,\ |x| > b \qquad ...(4.11)$$

$$F_c\,[\alpha\varphi(\xi) + \beta\psi(\xi); x] = f_0,\ a \leqslant |x| \leqslant b \qquad ...(4.12)$$

$$F_s\,[\beta\varphi(\xi) + \alpha\psi(\xi); x] = 0,\ a \leqslant |x| \leqslant b. \qquad ...(4.13)$$

From the results derived in section 2, we have

$$\left.\begin{aligned} X(z) &= [(z - a)(z + b)]^{-(1-2i\omega)/2}\,[(z + a)(z - b)]^{-(1+2i\omega)/2} \\ P(z) &= d_1 z + d_2,\ L(z) = f_0(z^2 + h_1 z + h_2) \\ \Lambda(z) &= if_0(2\beta)^{-1}\,[1 - (z^2 + d_1 z + d_2)\,X(z)] \\ f_0 &= k_1\,(k_2 + \Gamma)\,(1 + k_1\Gamma)\,p_0/\mu_1. \end{aligned}\right\} \qquad ...(4.14)$$

Hence

$$r(x) = f_0(\beta^2 - \alpha^2)^{-1/2}\,\{(x^2 - a^2)(b^2 - x^2)\}^{-1/2}\,[(x^2 + d_2')\sin \omega\theta - 2\omega(b - a)\cos \omega\theta],\quad a < x < b \qquad ...(4.15)$$

$$s(x) = -f_0(\beta^2 - \alpha^2)^{-1/2}\,\{(x^2 - a^2)(b^2 - x^2)\}^{-1/2}\,[(x^2 + d_2')\cos \omega\theta + 2\omega(b - a)\sin \omega\theta],\quad a < x < b \qquad ...(4.16)$$

where h_1, h_2 are known constants and $\theta = \log \{(x + b)(x - a)/(b - x)(x + a)\}$. If we write $d_1 = d_1' + id_1''$, $d_2 = d_2' + id_2''$, then it can be that

$$d_1' = d_2'' = 0,\ d_1'' = -2\omega(b - a).$$

The constant d_2' can be obtained from the condition (3.10). This condition leads to

$$\int_a^b [(x^2 + d_2')\cos \omega\theta + 2\omega(b - a)\sin \omega\theta]\,\{(b^2 - x^2)(x^2 - a^2)\}^{-1/2}\,dx = 0. \qquad ...(4.17)$$

This equation gives $d'_2 = -[I_2 + 2\omega(b-a)I_1]/I_0$

where $$I_0 = \int_a^b [(b^2 - x^2)(x^2 - a^2)]^{-1/2} \cos\omega\theta \, dx,$$

$$I_1 = \int_a^b [(b^2 - x^2)(x^2 - a^2)]^{-1/2} x \sin\omega\theta \, dx$$

$$I_2 = \int_a^b [(b^2 - x^2)(x^2 - a^2)]^{-1/2} x^2 \cos\omega\theta \, dx.$$

Lowengrub (1975) has mentioned that the constants d'_2 has to be calculated numerically. It is interesting to note that the above integrals can be evaluated in closed form. Hence the value of the constant d'_2 can be obtained in a closed form. These integrals are evaluated in the Appendix.

5. Three Collinear Cracks Opened by Constant Pressure

In this section, we shall consider the case when three cracks are located at the interface. The prescribed pressure $p(x) = p_0$, where p_0 is a constant. The three cracks are defined by $y = 0$, $-c < x < -b$, $-a < x < a$, $b < x < c$, where a, b, c are positive constants satisfying the condition $a < b < c$. For this problem from the results derived in section 2, we have

$$\left.\begin{aligned} X(z) &= \{(z^2 - a^2)(z^2 - b^2)(z^2 - c^2)\}^{-1/2} \\ &\quad \times \left\{\frac{(z+a)(z-b)(z+c)}{(z-a)(z+b)(z-c)}\right\}^{i\omega} \\ P(z) &= d_1 z^2 + d_2 z + d_3,\ L(z) = f_0(z^3 + h_1 z^2 + h_2 z + h_3) \end{aligned}\right\} \quad \ldots(5.1)$$

where h_1, h_2, h_3 are known constant and $f_0 = k_1(k_2 + \Gamma)(1 + k_1\Gamma)\, p_0/\mu_1$.

Hence from (2.16) we have

$$\Lambda(z) = f_0(2\beta)^{-1}[1 - (z^3 + d_1 z^2 + d_2 z + d_3)X(z)] \quad \ldots(5.2)$$

merging the known constants h_1, h_2, h_3 into arbitrary constants d_1, d_2, d_3.

Using (2.13), following values of $X^+(x)$ and $X^-(x)$ are obtained:

(i) for $b < x < c$

$$X^+(x) = -kX^-(x) = -ik^{1/2}(\cos\omega\theta_1 + i\sin\omega\theta_1) \times \{(x^2 - a^2)(x^2 - b^2)(c^2 - x^2)\}^{-1/2} \quad \ldots(5.3)$$

(ii) for $0 < x < a$

$$X^+(x) = -kX^-(x) = ik^{1/2}(\cos\omega\theta_1 + i\sin\omega\theta_1) \times \{(a^2 - x^2)(b^2 - x^2)(c^2 - x^2)\}^{-1/2} \quad \ldots(5.4)$$

(iii) for $a < x < b$

$$X^+(x) = X^-(x) = -(\cos\omega\theta_2 + i\sin\omega\theta_2) \times \{(x^2 - a^2)(b^2 - x^2)(c^2 - x^2)\}^{-1/2} \quad ...(5.5)$$

(iv) for $x > c$

$$X^+(x) = X^-(x) = (\cos\omega\theta_2 + i\sin\omega\theta_2) \times \{(x^2 - a^2)(x^2 - b^2)(x^2 - c^2)\}^{-1/2} \quad ...(5.6)$$

where $\theta_1 = \log\{(x + a)(b - x)(c + x)/(x - a)(b + x)(c - x)\}$

$$\theta_2 = \log\{(x + a)(x - b)(x + c)/(x - a)(x + b)(x - c)\}.$$

Hence

$$\lambda(x) = \Lambda^+(x) - \Lambda^-(x) = if_0(2\beta)^{-1}(x^3 + d_1x^2 + d_2x + d_3)[X^-(x) - X^+(x)]. \quad ...(5.7)$$

Let us write $d_j = d'_j + id''_j$, $j = 1, 2, 3$, where d'_j and d''_j are real constants. From the above equations and (2.7), (2.11) it can be easily demonstrated that

(i) for $b < x < c$

$$s(x) = -f_0(\beta^2 - \alpha^2)^{-1/2}\{(x^2 - a^2)(x^2 - b^2)(c^2 - x^2)\}^{-1/2} \times [(x^3 + d'_1x^2 + d'_2x + d'_3)\cos\omega\theta_1 - (d''_1x^2 + d''_2x + d''_3)\sin\omega\theta_1] \quad ...(5.8)$$

$$r(x) = f_0(\beta^2 - \alpha^2)^{-1/2}\{(x^2 - a^2)(x^2 - b^2)(c^2 - x^2)\}^{-1/2} \times [(x^3 + d'_1x^2 + d'_2x + d'_3)\sin\omega\theta_1 + (d''_1x^2 + d''_2x + d''_3)\cos\omega\theta_1) \quad ...(5.9)$$

(ii) for $0 < x < a$

$$s(x) = f_0(\beta^2 - \alpha^2)^{-1/2}\{(a^2 - x^2)(b^2 - x^2)(c^2 - x^2)\}^{-1/2} \times [(x^3 + d'_1x^2 + d'_2x + d'_3)\cos\omega\theta_2 - (d''_1x^2 + d''_2x + d''_3)\sin\omega\theta_2] \quad ...(5.10)$$

$$r(x) = -f_0(\beta^2 - \alpha^2)^{-1/2}\{(a^2 - x^2)(b^2 - x^2)(c^2 - x^2)\}^{-1/2} \times [(x^3 + d'_1x^2 + d'_2x + d'_3)\sin\omega\theta_2 + (d''_1x^2 + d''_2x + d''_3)\cos\omega\theta_2]. \quad ...(5.11)$$

Since $s(x)$ and $r(x)$ are odd and even functions of x, hence we must have

$$d'_1 = d'_3 = d''_2 = 0.$$

From (2.10), (2.7) and (5.2) for $a < x < b$, $x > c$ we have

$$\Lambda^+(x) = \Lambda^-(x) = \frac{1}{2\pi i}\int_L \frac{\lambda(u)}{u-x}\,du = \frac{1}{2\pi i}\int_L \frac{s(u)-ir(u)}{u-x}\,du$$

$$= if_0(2\beta)^{-1}\,[1-(x^3+d_1x^2+d_2x+d_3)\,X^+(x)]. \quad \ldots(5.12)$$

From (5.5), (5.6), (5.12), (3.8) and (3.9) we have, for $a < x < b$

$$\sigma_{yy}(x, 0+) = \sigma_{yy}(x, 0-) = p_0\,[1+\{(x^2-a^2)(b^2-x^2)(c^2-x^2)\}^{-1/2}$$
$$\times\,\{(x^3+d_2'x)\cos\omega\theta_1-(d_1''x^2+d_3'')\sin\omega\theta_1\}] \quad \ldots(5.13)$$

$$\sigma_{xy}(x, 0+) = \sigma_{xy}(x, 0-) = p_0\,\{(x^2-a^2)(b^2-x^2)(c^2-x^2)\}^{-1/2}$$
$$\times\,[(x^3+d_2'x)\sin\omega\theta_1+(d_1''x^2+d_3'')\cos\omega\theta_1]. \quad \ldots(5.14)$$

Similarly for $x > c$, we have

$$\sigma_{yy}(x, 0+) = \sigma_{yy}(x, 0-) = p_0\,[1-\{(x^2-a^2)(x^2-b^2)(x^2-c^2)\}^{-1/2}$$
$$\times\,\{(x^3+d_2'x)\cos\omega\theta_2-(d_1''x^2+d_3'')\sin\omega\theta_2\}] \quad \ldots(5.15)$$

$$\sigma_{xy}(x, 0+) = \sigma_{xy}(x, 0-) = -\,p_0\,\{(x^2-a^2)(x^2-b^2)(x^2-c^2)\}^{-1/2}$$
$$\times\,[(x^3+d_2'x)\sin\omega\theta_2+(d_1''x^2+d_3'')\cos\omega\theta_2]. \quad \ldots(5.16)$$

The above expressions for stress components show that at the edges of the cracks at $x = a$, $x = b$, $x = c$ violent oscillations occur. From these equations, for large values of 'x', we have

$$\sigma_{yy}(x, 0+) = O(x^{-2}),\ \sigma_{xy}(x, 0+) = p_0\,\{2\omega(a+c-b)+d_1''\}\,x^{-1}+O(x^{-2}).$$

Since the stress components for large 'x' are of the order $O(x^{-1})$ we must have

$$d_1'' = -2\omega(a+c-b).$$

The remaining constants d_2' and d_3'' can be obtained from conditions (3.10). From this condition we have

$$\int_b^c \frac{(x^3+id_1''x^2+d_2'x+id_3'')(\cos\omega\theta+i\sin\omega\theta)}{\{(x^2-a^2)(x^2-b^2)(c^2-x^2)\}^{1/2}}\,dx = 0 \quad \ldots(5.17)$$

or,

$$[b(b+c)\,id_1''+bd_2'+b(b+c)^2+id_3'']\,J_1+[(b+c)\,id_1''+d_2'$$
$$+\,b^2+bc+c^2]\,J_2-(c+2b+id_1'')\,J_3-J_4 = 0 \quad \ldots(5.18)$$

where

$$J_1 = \int_b^c F(x)\,dx,\quad J_2 = \int_b^c (x-b)\,F(x)\,dx,\quad J_3 = \int_b^c (x-b)(c-x)\,F(x)\,dx$$

$$J_4 = \int_b^c (x - b)^2 (c - x) F(x)\, dx$$

$$F(x) = (x^2 - a^2)(x^2 - b^2)(c^2 - x^2)^{-1/2} \left\{\frac{(x + a)(x + c)(x - b)}{(x - a)(x + b)(c - x)}\right\}^{i\omega}.$$

The integrals J_1, J_2, J_3, J_4 have been evaluated in the appendix. On separating the real and imaginary parts the values of unknown constants d_2' and d_3'' can be obtained. Calculations have been made for a particular case when $a = 0.2$, $b = 0.6$, $c = 1.0$, $\eta_1 = 0.22$, $\eta_2 = 0.35$, $E_1 = 10^7$ psi, $E_2 = 4.5 \times 10^5$ psi. For these values we have $d_2' = 0.1136$, $d_3'' = 0.0009$.

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APPENDIX

We shall now evaluate the integrals I_0, I_1, I_2 which occur in section 4. These integrals are obtained by separating the real and imaginary parts of the following integrals which have been evaluated by making the substitution $x = a\cos^2\theta + b\sin^2\theta$ and by using the binomial expansion:

$$\int_a^b \frac{\exp(i\omega\theta)\, dx}{[(x^2 - a^2)(b^2 - x^2)]^{1/2}} = \int_a^b (x - a)^{-(1-2i\omega)/2} (x + a)^{-(1+2i\omega)/2} \times (b - x)^{-(1+2i\omega)/2} (b + x)^{-(1+2i\omega)/2}\, dx$$

$$= \frac{\pi}{(a + b)\cosh \omega\pi} F_3 [\tfrac{1}{2} + i\omega, \tfrac{1}{2} - i\omega, \tfrac{1}{2} - i\omega, \tfrac{1}{2} + i\omega; 1; z, -z] \quad \text{...(A1)}$$

$$\int_a^b \{(x^2 - a^2)(b^2 - x^2)\}^{-1/2} x \exp(i\omega\theta)\, dx$$

$$= z\,\Gamma(\tfrac{1}{2} - i\omega)\,\Gamma(\tfrac{3}{2} + i\omega)\, F_3 [\tfrac{1}{2} + i\omega, \tfrac{1}{2} - i\omega, \tfrac{1}{2} - i\omega, \tfrac{3}{2} + i\omega; 2; z, -z]$$

$$+ \frac{a}{a + b}\Gamma(\tfrac{1}{2} - i\omega)\,\Gamma(\tfrac{1}{2} + i\omega)\, F_3 [\tfrac{1}{2} + i\omega, \tfrac{1}{2} - i\omega, \tfrac{1}{2} - i\omega, \tfrac{1}{2} + i\omega; 1; z, -z] \quad \text{...(A2)}$$

$$\int_a^b \{(x^2 - a^2)(b^2 - x^2)\}^{-1/2} x^2 \exp(i\omega\theta)\, dx$$

$$= \frac{\pi a^2}{(a+b)\cosh \pi\omega} F_3 [\tfrac{1}{2} + i\omega, \tfrac{1}{2} - i\omega, \tfrac{1}{2} - i\omega, \tfrac{1}{2} + i\omega; 1; z, -z]$$

$$+ (b-a)\Gamma(\tfrac{1}{2}-i\omega)\Gamma(\tfrac{3}{2}+i\omega) F_3 [\tfrac{1}{2}+i\omega, \tfrac{1}{2}-i\omega, \tfrac{1}{2}-i\omega; \tfrac{3}{2}+i\omega; 2; z, -z]$$

$$- \frac{(b-a)z}{b}\Gamma(\tfrac{3}{2}+i\omega)\Gamma(\tfrac{3}{2}-i\omega) F_3 [\tfrac{1}{2}+i\omega, \tfrac{1}{2}-i\omega, \tfrac{3}{2}-i\omega;$$

$$\tfrac{3}{2}+i\omega; 3; z, -z] \quad \text{...(A3)}$$

where $z = (b-a)/(b+a)$ and F_3 is hypergeometric function of two variables defined in Gradshteyn and Ryzhik (1965, p.1053). On separating real and imaginary parts, we get

$$I_0 = (a+b)^{-1} \pi \operatorname{sech} \pi\omega \; {}_2F_1 [\tfrac{1}{2} + i\omega, \tfrac{1}{2} - i\omega; 1; z^2] \quad \text{...(A4)}$$

$$I_1 = \pi\omega z \operatorname{sech} \pi\omega \, F_3 [\tfrac{1}{2} + i\omega, \tfrac{1}{2} - i\omega, \tfrac{1}{2} - i\omega, \tfrac{1}{2} + i\omega; 2; z, -z] \quad \text{...(A5)}$$

$$I_2 = \pi a^2 (a+b)^{-1} \operatorname{sech} \pi\omega \, F_3 [\tfrac{1}{2} + i\omega, \tfrac{1}{2} - i\omega, \tfrac{1}{2} - i\omega, \tfrac{1}{2} + i\omega; 1, z, -z]$$

$$+ \frac{\pi}{2}(b-a) \operatorname{sech} \pi\omega \, F_3 [\tfrac{1}{2} + i\omega, \tfrac{1}{2} - i\omega, \tfrac{1}{2} - i\omega, \tfrac{1}{2} + i\omega; 2; z, -z]$$

$$- \frac{\pi}{4} z(b-a) \operatorname{sech} \pi\omega$$

$$\times \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(\tfrac{1}{2}+i\omega)_p(\tfrac{1}{2}-i\omega)_p(\tfrac{1}{2}-i\omega)_q(\tfrac{1}{2}+i\omega)_q(2q+1)(q-p)}{(p)!\,(q)!(3)_{p+q}}$$

$$\times z^p(-z)^q. \quad \text{...(A6)}$$

It may be noted that the above series are rapidly convergent. For any numerical computation of the integrals it is sufficient to consider only a first few terms of above series.

The integrals occurring in section 5 can be evaluated by substituting

$$x = b\cos^2\theta + c\sin^2\theta.$$

Binomial expansion of the integrand in powes of $\sin\theta$ and $\cos\theta$ give

$$J_1 = KF_D [\tfrac{1}{2} - i\omega, \tfrac{1}{2} - i\omega, \tfrac{1}{2} + i\omega, \tfrac{1}{2} + i\omega, \tfrac{1}{2} - i\omega; 1, z_1, z_2, z_3, z_4] \quad \text{...(A7)}$$

$$J_2 = K(c-b)(\tfrac{1}{2} + i\omega) F_D [\tfrac{1}{2} - i\omega, \tfrac{1}{2} - i\omega, \tfrac{1}{2} + i\omega, \tfrac{1}{2} + i\omega,$$

$$\tfrac{1}{2} - i\omega; 2; z_1, z_2, z_3, z_4] \quad \text{...(A8)}$$

$$J_3 = \tfrac{1}{2}(c-b)^2\,(\tfrac{1}{4}+\omega^2)\,KF_D\,[\tfrac{3}{2}-i\omega, \tfrac{1}{2}-i\omega, \tfrac{1}{2}+i\omega, \tfrac{1}{2}+i\omega, \tfrac{1}{2}-i\omega; 3; z_1, z_2, z_3, z_4] \qquad \text{...(A9)}$$

$$J_4 = (c-b)^3\,(\tfrac{1}{4}+\omega^2)\,(\tfrac{3}{2}+i\omega)\,KF_2\,[\tfrac{3}{2}-i\omega, \tfrac{1}{2}-i\omega, \tfrac{1}{2}+i\omega, \tfrac{1}{2}+i\omega, \tfrac{1}{2}-i\omega; 4; z_1, z_2, z_3, z_4] \qquad \text{...(A10)}$$

where

$$F_D\,[a; b_1, b_2, \ldots, b_n; c; z_1, z_2, \ldots, z_n] = \sum_{m_1, m_2 \ldots m_n = 0}^{\infty} \frac{(a)_{m_1+m_2+\ldots+m_n}\,(b_1)_{m_1}\,(b_2)_{m_2}\ldots(b_n)_{m_n}\,z_1^{m_1}\ldots z_n^{m_n}}{(m_1)!\,(m_2)!\ldots(m_n)!\,(c)_{m_1+m_2+\ldots+m_n}}$$

$$z_1 = (c-b)/(c+a),\; z_2 = (c-b)/(c-a),\; z_2 = (c-b)/(c+b),$$

$$z_4 = (c-b)/2c,$$

$$K = (\pi/\cosh \pi\omega)\,\{2c(c+b)\,(c^2-a^2)\}^{-1/2}\,\{2c(c+a)/(c+b)\,(c-a)\}^{i\omega}.$$

The function F_D can be approximated by the first few terms since z_1, z_2, z_3, z_4 are less than unity and hence the series is rapidly convergent.