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CONVERGENCE OF THE LAGRANGE INTERPOLATION ON THE EXTENDED TCHEBYCHEFF NODES

VIRENDRA KUMAR AND K. K. MATHUR

Department of Mathematics and Astronomy, Lucknow University, Lucknow 226001

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According to the classical results of Faber (1914) and Bernstein (1931) the sequence of Lagrange interpolation processes $\{L_n(f, x)\}_{n=1}^{\infty}$ is not convergent for continuous function $f(x)$ in $[-1, 1]$. Later on Bermán (1965) showed that $L_n(f, x)$ diverges at $x = 0$ even for $f(x) = |x|$. These results inspired us to prove the following :

Let $f(x) = (1 - x^2)\, g(x)$, where $g(x)$ is a continuous function in $[-1, 1]$. Then the operator

$$\tfrac{1}{2}\left[L_n\left(f, \theta - \frac{\pi}{2n}\right) + L_n\left(f, \theta + \frac{\pi}{2n}\right)\right], \quad x = \cos\theta$$

constructed on the knots

$$x_0 = 1,\ x_k = \cos\left(\frac{2k-1}{2n}\right)\pi,\ \ k = 1, 2, \ldots, n;\ x_{n+1} = -1$$

converges uniformly to $f(\cos\theta)$ in $[-1, 1]$.

§1. Let

$$1 = x_0 > x_1 > \ldots > x_n > x_{n+1} = -1 \qquad \ldots(1)$$

be $n + 2$ distinct points. Then the Lagrange interpolation process, which assumes the values $f(x_0), f(x_1), \ldots, f(x_n), f(x_{n+1})$ at the abscissas in (1), has the form

$$L_n(f, x) = L_n(f, \theta) = \cos^2\theta/2.\, f(1)\, \frac{W_n(\cos\theta)}{W_n(1)} + \sin^2\theta/2.\, f(-1)\, \frac{W_n(\cos\theta)}{W_n(-1)}$$

$$+ \sum_{k=1}^{n} f(\cos\theta_k)\, \frac{\sin^2\theta}{\sin^2\theta_k}\, l_k(\theta),\ x = \cos\theta,\ x_k = \cos\theta_k \qquad \ldots(2)$$

where

$$l_k(\theta) = \frac{W_n(\cos\theta)}{W'_n(\cos\theta_k)\,(\cos\theta - \cos\theta_k)} \qquad \ldots(3)$$

and

$$W_n(\cos\theta) = \prod_{k=1}^{n} (\cos\theta - \cos\theta_k). \qquad \ldots(4)$$

The sequence $\{L_n(f, x)\}_{n=1}^{\infty}$ [according to the classical theorem of Faber (1914) and Bernstein (1931)] is not convergent for every continuous function $f(x)$ in $[-1, 1]$. Further Bermán (1965) showed that $L_n(f, x)$ for the function $f(x) = |x|$ diverges at $x = 0$, when

$$x_k = \cos\left(\frac{2k-1}{2n}\right)\pi, \; k = 1, 2, ..., n$$

are the roots of nth Tchebycheff polynomial of the first kind i.e. $T_n(x) = \cos n\theta$, $x = \cos\theta$.

In this paper we prove the following :

Theorem — Let $g(x)$ be a continuous function in $[-1, 1]$ and $f(x) = (1 - x^2) g(x)$, then the operator

$$V_n(f, x) = V_n(f, \theta) = \tfrac{1}{2}\,[L_n(f, \theta - (\pi/2n)) + L_n(f, \theta + (\pi/2n))], \; x = \cos\theta$$

converges uniformly to $f(\cos\theta)$ in $[-1, 1]$, when $x_k = \cos\left(\frac{2k-1}{2n}\right)\pi$; $k = 1, ... n$.

§2. The Lagrange interpolation polynomial (2) constructed for

$$f(x) = (1 - x^2)\, g(x), \text{ when } x_k = \cos\left(\frac{2k-1}{2n}\right)\pi, \; k = 1, 2, ..., n$$

is given by

$$L_n(f, \theta) = \sum_{k=1}^{n} g(\cos\theta_k)\, A_k(\theta) \qquad ...(5)$$

where

$$A_k(\theta) = \sin^2\theta\, l_k(\theta), \; k = 1, ..., n \qquad ...(6)$$

and

$$l_k(\theta) = \frac{(-)^{k+1}\cos n\theta \sin\theta_k}{n\,(\cos\theta - \cos\theta_k)}, \; k = 1, ..., n \qquad ...(7)$$

are the fundamental polynomials of Lagrange interpolation constructed on the roots of $T_n(x) = \cos n\theta$, $x = \cos\theta$.

From (6), we obtain the identity

$$\sum_{k=1}^{n} A_k(\theta) = \sin^2\theta. \qquad ...(8)$$

Also

$$\tfrac{1}{2}\,|\,A_k(\theta - (\pi/2n)) + A_k(\theta + (\pi/2n))\,| \leqslant \frac{\pi^3}{4n^2(\theta - \theta_k - (\pi/2n))^2} \qquad ...(9)$$

owing to (6), the Abel's inequality and the inequality (Grünwald 1941)

$$\tfrac{1}{2} \mid l_k(\theta - (\pi/2n)) + l_k(\theta + (\pi/2n)) \mid \leqslant \frac{\pi^3}{4n^2(\theta - \theta_k - (\pi/2nd))^2}.$$

Further, one can easily prove the following lemmas applying the method of Grünwald (1941) with the help of (9).

Lemma 1 — If $0 \leqslant \theta \leqslant \pi$, then

$$\tfrac{1}{2} \sum_{k=1}^{n} \mid A_k(\theta - (\pi/2n)) + A_k(\theta + (\pi/2n)) \mid \leqslant C, \qquad \ldots(10)$$

where $C > 0$ is an absolute constant.

Lemma 2 — For sufficiently large n and fixed $\delta > 0$,

$$\sum_{1 \leqslant k \leqslant n,\ \mid \theta - \theta_k \mid > \delta} \tfrac{1}{2} \mid A_k(\theta - (\pi/2n)) + A_k(\theta + (\pi/2n)) \mid = O(1/n). \qquad \ldots(11)$$

§3. *Proof of the theorem* — From the identity (6), we have

$$\tfrac{1}{2} \sum_{k=1}^{n} [A_k(\theta - (\pi/2n)) + A_k(\theta + (\pi/2n))]$$

$$= \tfrac{1}{2} [\sin^2 (\theta - (\pi/2n)) + \sin^2 (\theta + (\pi/2n))].$$

Hence, for $f(x) = (1 - x^2)\, g(x)$,

$$\begin{aligned} \mid V_n(f, \theta) - f(\cos \theta) \mid \leqslant & \mid [V_n(f, \theta) - \tfrac{1}{2} \{\sin^2 (\theta - (\pi/2n)) \\ & + \sin^2 (\theta + (\pi/2n))\}\, g(\cos \theta)] \mid \\ & + \mid [\tfrac{1}{2} \{\sin^2 (\theta - (\pi/2n)) + \sin^2 (\theta + (\pi/2n))\} \\ & \times g(\cos \theta) - \sin^2\theta\, g(\cos \theta)] \mid = I_1 + I_2. \end{aligned} \qquad \ldots(12)$$

Now

$$I_1 \leqslant \sum_{k=1}^{n} \tfrac{1}{2} \mid g(\cos \theta_k) - g(\cos \theta) \| A_k(\theta - (\pi/2n)) + A_k(\theta + (\pi/2n)) \mid$$

$$= \sum_{(\theta - \theta_k) \leqslant \delta} + \sum_{(\theta - \theta_k) > \delta} \equiv \Sigma_1 + \Sigma_2.$$

From (10) and the continuity of $g(x)$, we have

$$\Sigma_1 \leqslant \epsilon/2. \qquad \ldots(13)$$

For Σ_2, we have from (11)

$$\Sigma_2 = M\, O(1/n), \qquad \text{...(14)}$$

where $\quad M = \max_{-1 \leqslant x \leqslant 1} |\, g(x) \,|$.

Further, one can easily see that

$$\begin{aligned} I_2 &\leqslant \tfrac{1}{2} [\, |\sin^2 (\theta - (\pi/2n)) - \sin^2 \theta \,| \\ &\quad + |\sin^2 (\theta + (\pi/2n)) - \sin^2 \theta \,|\,]\, |\, g(\cos \theta) \,| \\ &\leqslant \frac{M\pi}{n}. \end{aligned} \qquad \text{...(15)}$$

Combining the inequalities (13), (14), (15) and (12), we complete the proof of the theorem.

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