Indian J. pure appl. Math., **10**(6) : 699-703, June 1979

ON SEQUENCES OF MAPPINGS AND FIXED POINTS IN UNIFORM SPACES II

S. N. MISHRA

Department of Mathematics, University of Allahabad, Allahabad

(*Received* 14 *June* 1978; *after revision* 4 *January* 1979)

Some fixed point theorems for sequences of mappings and their fixed points in uniform spaces are proved and the results obtained are the generalizations of the corresponding results for metric spaces.

1. INTRODUCTION

Several mathematicians have investigated the conditions under which the convergence of a squence of mappings $T_n(n = 1, 2, ...)$ to a mapping T_0 of a metric space into itself implies the convergence of the scquence of their fixed points $a_n(n = 1, 2, ...)$ to the fixed point a_0 of T_0. For related results one is referred to Bonsall (1962), Nadler (1968), Fraser and Nadler (1969), Singh (1969), Reich (1971), Hardy and Rogers (1973) and Iseki (1974). In the present note we prove some convergence theorems on fixed points in uniform spaces and the results obtained generalize some of the recent results of Hardy and Rogers (1973) and Iseki (1974) (for metric spaces) to uniform spaces.

2. TOPOLOGICAL PRELIMINERIES

Throughout the discussion X stands for a sequentially complete Hausdorff uniform space defined by a family P of pseudometrics on X. Let us put

$$V_{(\rho,r)} = \{(x, y) : x, y \in X, \rho(x, y) < r, r > 0\}$$

$$G = \{V : V = \bigcap_{i=1}^{n} V_{(\rho_i,r_i)}, \rho_i \in P, r_i > 0, i = 1, 2, ..., n\}$$

For $V = \bigcap_{i=1}^{n} V_{(\rho_i,r_i)} \in G$, let

$$\alpha V = \bigcap_{i=1}^{n} V_{(\rho_i,\alpha r_i)}, \quad \alpha > 0$$

$$= \triangle \text{ (the diagonal)}, \quad \alpha = 0$$

We use the following well-known lemmas (cf. Acharya 1974).

Lemma 2.1 — If $V \in G$ and ρ is any pseudometric on X such that for $\alpha, \beta > 0$,

$$(x, y) \in V_{(\rho, \alpha r_1)} \circ V_{(\rho, \beta r_2)}$$

then

$$\rho(x, y) < \alpha r_1 + \beta r_2.$$

Lemma 2.2 — If $V \in G$ is arbitrary, then there is a pseudometric p on X such that

$$V = V_{(p,1)}.$$

This p is called a Minkowski's pseudometric of V.

3. Convergence Theorems

In this section we prove the following theorems.

Theorem 3.1 — Let T_n be a mapping of X into itself with atleast one fixed point a_n for each $n = 1, 2, 3, \ldots$. Suppose there are nonnegative real numbers α_i $(i = 1, 2, \ldots, 5)$ with $\alpha_3 + \alpha_4 + \alpha_5 < 1$ such that for $V_i \in G$ $(i = 1, 2, \ldots, 5)$ and $x, y \in X$,

$$(T_n x, T_n y) \in \alpha_1 V_1 \circ \alpha_2 V_2 \circ \alpha_3 V_3 \circ \alpha_4 V_4 \circ \alpha_5 V_5 \qquad \ldots(1)$$

if $(x, T_n x) \in V_1$, $(y, T_n y) \in V_2$, $(x, T_n y) \in V_3$, $(y, T_n x) \in V_4$ and $(x, y) \in V_5$. If the sequence $\{T_n\}_{n=1}^{\infty}$ converges pointwise to a mapping $T_0 : X \to X$ with fixed point a_0, then the sequence $\{a_n\}_{n=1}^{\infty}$ converges to a_0.

Proof : Let $V \in G$ be arbitrary. Denote by p a Minkowski's pseudometric of V so that $V = V_{(p,1)}$. Let $x, y \in X$. We write $p(x, T_n x) = s_1$, $p(y\, T_n y) = s_2$, $p(x, T_n y) = s_3$, $p(y, T_n x) = s_4$ and $p(x, y) = s_5$. Take $\epsilon > 0$. Then

$$(x, T_n x) \in V_{(p, s_1+\epsilon)},\ (y, T_n y) \in V_{(p, s_2+\epsilon)},\ (x, T_n y) \in V_{(p, s_3+\epsilon)},$$

$$(y, T_n x) \in V_{(p, s_4+\epsilon)}, \text{ and } (x, y) \in V_{(p, s_5+\epsilon)}.$$

Using condition (1), Lemma 2.1 and a routine calculation (cf. Mishra 1978) we get

$$p(T_n x, T_n y) \leqslant \alpha_1 p(x, T_n x) + \alpha_2 p(y, T_n y) + \alpha_3 p(x, T_n y) + \alpha_4 p(y, T_n x) + \alpha_5 p(x, y). \qquad \ldots(2)$$

Now,

$$p(a_n, a_0) = p(T_n a_n, T_0 a_0) \leqslant p(T_n a_n, T_n a_0) + p(T_n a_0, T_0 a_0).$$

Using condition (2) and the fact that $p(T_n a_n, a_n) = 0$, we get

$$p(a_n, a_0) \leqslant \left(\frac{1 + \alpha_2 + \alpha_3}{1 - \alpha_3 - \alpha_4 - \alpha_5}\right) p(T_n a_0, T_0 a_0). \qquad ...(3)$$

Since $\{T_n\}_{n=1}^{\infty}$ converges pointwise to T_0, for $\epsilon > 0$ and a point $a_0 \in X$, there is an N such that for all $n \geqslant N$ we have

$$p(T_n a_0, T_0 a_0) < \left(\frac{1 - \alpha_3 - \alpha_4 - \alpha_5}{1 + \alpha_2 + \alpha_3}\right) \epsilon. \qquad ...(4)$$

From (3) and (4) we get

$$p(a_n, a_0) < \epsilon.$$

This shows that $(a_n, a_0) \in V$. Since V is arbitrary and X is Hausdorff, it follows that $\{a_n\}_{n=1}^{\infty}$ converges to a_0. This completes the proof.

Example — Let $X = [0, 2]$ with the usual uniformity. Let us define $T_n : [0, 2] \to [0, 2]$ as

$$T_n(x) = 1 + \frac{x}{2n + 2} \text{ for all } x \in [0, 2], \; n = 1, 2, \ldots .$$

Then the fixed point of T_n is given by

$$a_n = \frac{2n + 2}{2n + 1} \text{ for each } n = 1, 2, \ldots$$

Now, $\lim_{n \to \infty} a_n = 1$. Also, $T_0(x) = \lim_{n \to \infty} T_n(x) = 1$ and 1 is the only fixed point of T_0. Therefore the conclusion of theorem 3.1 holds. It can be easily seen that with the proper choice of the constants $\alpha_i (i = 1, 2, \ldots, 5)$, T_n satisfies the condition (1).

Remark : It is not necessary to take $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$ as assumed by Hardy and Rogers (1973) and Iseki (1974). All that we need is $\alpha_3 + \alpha_4 + \alpha_5 < 1$ (The existence of a fixed point being assumed).

Theorem 3.2 — Let T_n be a mapping of X into itself with atleast one fixed point a_n for each $n = 1, 2, \ldots$. Let $T_0 : X \to X$ be a mapping with fixed point a_0 such that for $V_i \in G (i = 1, 2, \ldots, 5)$ and $x, y \in X$,

$$(T_0 x, T_0 y) \in \alpha_1 V_1 \circ \alpha_2 V_2 \circ \alpha_3 V_3 \circ \alpha_4 V_4 \circ \alpha_5 V_5 \qquad ...(5)$$

if $(x, T_0 x) \in V_1$, $(y, T_0 y) \in V_2$, $(x, T_0 y) \in V_3$, $(y, T_0 x) \in V_4$ and $(x, y) \in V_5$ where $\alpha_i \geqslant 0 (i = 1, 2, \ldots, 5)$ with $\alpha_3 + \alpha_4 + \alpha_5 < 1$. If the sequence $\{T_n\}_{n=1}^{\infty}$ converges uniformly to T_0, then the sequence $\{a_n\}_{n=1}^{\infty}$ converges to a_0.

PROOF : Let $V \in G$ be arbitrary and p be a Minkowski's pseudometric of V. Take $x, y \in X$. Then using the proof technique of theorem 3.1, it can be shown that

$$p(T_0x, T_0y) \leqslant \alpha_1 p(x, T_0x) + \alpha_2 p(y, T_0y) + \alpha_3 p(x, T_0y) + \alpha_4 p(y, T_0x) + \alpha_5 p(x, y). \quad ...(6)$$

We have

$$p(a_n, a_0) = p(T_n a_n, T_0 a_0) \leqslant p(T_n a_n, T_0 a_n) + p(T_0 a_n, T_0 a_0).$$

Using (6) and a routine calculation it can be shown that

$$p(a_n, a_0) \leqslant \left(\frac{1 + \alpha_1 + \alpha_4}{1 - \alpha_3 - \alpha_4 - \alpha_5}\right) p(T_n a_n, T_0 a_n) \quad ...(7)$$

Since $\{T_n\}_{n=1}^{\infty}$ converges uniformly to T_0, for $\epsilon > 0$ there is an N such that for all $n \geqslant N$ we have

$$p(T_n a_n, T_0 a_n) < \left(\frac{1 - \alpha_3 - \alpha_4 - \alpha_5}{1 + \alpha_1 + \alpha_4}\right) \epsilon. \quad ...(8)$$

Using (7), (8) and the arguments of the previous theorem, it follows that $\{a_n\}_{n=1}^{\infty}$ converges to a_0. This completes the proof.

Example — Let $X = [0, 2]$ with the usual uniformity. Define $T_n : [0, 2] \to [0, 2]$ as

$$T_n(x) = \frac{1}{n} + \left(\frac{n}{2n+1}\right) x, \text{ for all } x \in [0, 2] \quad (n = 1, 2, \ldots).$$

Then the fixed point of T_n is given by

$$a_n = \frac{2n+1}{n(n+1)} \text{ for each } n = 1, 2, \ldots .$$

Now, $\lim_{n\to\infty} a_n = 0$. Also, $T_0(x) = \lim_{n\to\infty} T_n(x) = \frac{1}{2} x$ and $x = 0$ is the unique fixed point of T_0. Therefore the conclusion of theorem 3.2 holds.

ACKNOWLEDGEMENT

The author wishes to express his sincere thanks to Dr. (Mrs.) Pramila Srivastava for her kind suggestions. Thanks are also due to Professor Kiyoshi Iseki (Kobe University) for providing useful information and the referee for his valuable comments.

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