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UNSTEADY FLOW OF A DUSTY GAS IN A CHANNEL WHOSE CROSS-SECTION IS AN ANNULAR SECTOR

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In this paper, the unsteady flow of a viscous incompressible gas embedded with small spherical particles in a channel whose cross section is an annular sector is discussed. Initially the gas and particles are at rest and flow takes place under the influence of an arbitrary time varying pressure gradient. Explicit expressions for the exact velocities of the gas and particles are obtained by using the operational methods. Effects of mass concentration and relaxation time of particles on the velocities are discussed in detail.

INTRODUCTION

The unsteady flow of a viscous incompressible fluid through channels under a time varying pressure gradient has been considered by several investigators (Mithal 1960, Mittal 1962, Prakash 1969, Rawat 1970). The addition of particles to this body of fluid renders the problems more complex but, nevertheless, not intractable. These solutions are expected to contribute to the understanding of the boundary layer in such a fluid, as did the solution of the classical problems.

Michael and Miller (1966) and Liu (1966, 1967) have studied the flow produced by the motion of an infinite plate in a dusty gas occupying the semi-infinite space above it. Later Healy and Yang (1972) have obtained exact solutions for the problems discussed by the above authors, using the technique of Laplace transform. Recently, Gupta and Gupta (1976) have studied the flow of a dusty gas in a rectangular channel under the influence of an arbitrary time varying pressure gradient. Gupta and Gupta (1977) have discussed the flow of the dusty gas in the annular space between two concentric cylinders when both the cylinders are moving with arbitrary time dependent velocities.

The present communication reports the flow of a viscous incompressible gas embedded with small spherical particles in a channel whose cross-section is an annular sector. The purpose of this paper is to develop the general time dependent flow model and to obtain explicit expressions for both the gas and particles velocities, when the pressure gradient is an arbitrary function of time, in exact form. The flow when the pressure gradient is an absolute constant has been discussed as a particular case of the main results. It has been found that the change of mass concentration of particles has more effects on the velocities than that of the relaxation time.

Non-circular ducts are frequently used in automobiles radiators, nuclear power plants, aerospace vehicles etc., as they can be fitted within the available space between compactly placed components. The study of flow of fluid embedded with particles in such ducts is of great importance, because the behaviour of oil or fuel in the ducts due to the presence of particles is considerably changed and many new phenomena are observed.

Formulation of the Problem

Saffman (1962) has presented differential equations of motion for a dusty viscous incompressible gas, in which the dust particles are uniformly distributed. The derivation presented here is based upon Saffman's model. In this paper, we have considered the flow of the dusty gas in a channel whose cross-section is a sector bounded by two radii $\theta = \pm \alpha$ and the circles $R = a$ and $R = b$ (Fig. 1 represents the geometry and coordinate system). The present investigation assumes that the particles are spherical and uniform in size, and the bulk concentration (concentration by volume) of dust is very small. However, the mass concentration of dust can be of order unity by allowing the ratio of the density of the dust and gas to be large. Following Saffman it is assumed that the steady Stokes' law of resistance between the particles and gas is applicable.

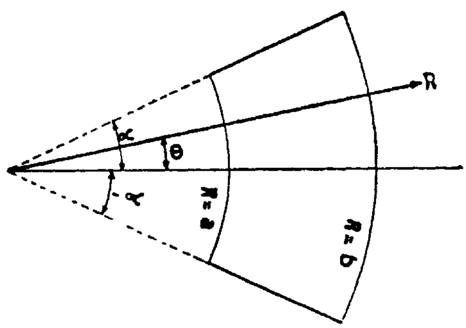

FIG. 1. Geometry of cross-section and coordinate system.

In cylindrical polar coordinate (R, θ, Z), the appropriate equations of motion from Saffman (1962) and Gupta and Gupta (1976) appear in the form

$$\frac{\partial u_z}{\partial T} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 u_z}{\partial R^2} + \frac{1}{R}\frac{\partial u_z}{\partial R} + \frac{1}{R^2}\frac{\partial^2 u_z}{\partial \theta^2}\right) + \frac{KN_0}{\rho}(v_z - u_z) \quad \text{...(1)}$$

and

$$\frac{\partial v_z}{\partial T} = \frac{K}{m}(u_z - v_z) \quad \text{...(2)}$$

where u_z, v_z are the axial components of the velocities of the gas and particles respectively, m is the mass of a particle, N_0 the number density of the particles which is constant throughout the motion, K the Stokes' resistance coefficient, p, T, ν and ρ are the pressure, time, kinematic viscosity and density of the gas respectively.

Introducing the following dimensionless quantities

$$u = au_z/\nu,\ v = av_z/\nu,\ \bar{z} = Z/a,\ r = R/a,\ \bar{p} = (a^2/\rho\nu^2)\,p,\ t = T\nu/a^2$$

eqns. (1) and (2) become

$$\frac{\partial u}{\partial t} = -\frac{\partial \bar{p}}{\partial \bar{z}} + \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) + \beta(v - u) \qquad \ldots(3)$$

and

$$\frac{\partial v}{\partial t} = \gamma'(u - v) \qquad \ldots(4)$$

where $f = N_0 m/\rho$ is the mass concentration of particles, $\gamma = (m/k)/(a^2/\nu)$ is the dimensionless relaxation time of particles,

$$\beta = f/\gamma = N_0 K a^2/\rho\nu \text{ and } \gamma' = 1/\gamma.$$

The initial and boundary conditions for the problem are :

Initial conditions :

$$t \leqslant 0 \qquad \left.\begin{aligned} u(r, \theta, t) &= 0 \\ v(r, \theta, t) &= 0 \end{aligned}\right\} \text{ for } -\alpha \leqslant \theta \leqslant \alpha,\ 1 \leqslant r \leqslant \sigma \qquad \ldots(5)$$

Boundary conditions :

$$t > 0 \qquad \left.\begin{aligned} u(r, \pm\alpha, t) &= 0 \\ v(r, \pm\alpha, t) &= 0 \end{aligned}\right\} \text{ for } 1 \leqslant r \leqslant \sigma \qquad \ldots(6)$$

$$\left.\begin{aligned} u(1, \theta, t) &= 0 \\ v(1, \theta, t) &= 0 \end{aligned}\right\} \text{ for } -\alpha \leqslant \theta \leqslant \alpha \qquad \ldots(7)$$

$$\left.\begin{aligned} u(\sigma, \theta, t) &= 0 \\ v(\sigma, \theta, t) &= 0 \end{aligned}\right\} \text{ for } -\alpha \leqslant \theta \leqslant \alpha \qquad \ldots(8)$$

where $\quad \sigma = b/a > 1.$

From the boundary conditions (6) it is evident that the flow is symmetrical about the plane $\theta = 0$, therefore, the flow in the region $0 \leqslant \theta \leqslant \alpha$ is considered and accordingly the boundary conditions (6) are equivalent to

and

$$\left.\begin{array}{l}\left.\begin{array}{l}u(r, \alpha, t) = 0 \\ v(r, \alpha, t) = 0\end{array}\right\} \text{ for } 1 \leqslant r \leqslant \sigma \\ \\ \dfrac{\partial u}{\partial \theta} = 0, \dfrac{\partial v}{\partial \theta} = 0 \text{ for } \theta = 0.\end{array}\right\} \quad \ldots(9)$$

SOLUTION OF THE PROBLEM

We shall solve the problem by using the methods of operational calculus. Using first the finite cosine transform with respect to θ defined by

$$\bar{u}(r, n, t) = \int_0^\alpha u(r, \theta, t) \cos q_n\theta \, d\theta \quad \ldots(10)$$

where

$$q_n = \frac{(2n+1)\pi}{2\alpha} \quad \ldots(11)$$

and then the finite Hankel transform with respect to r defined by

$$\bar{u}_H(p_i, n, t) = \int_1^\sigma \bar{u}(r, n, t) \, . \, rB_{q_n}(p_i r) \, dr \quad \ldots(12)$$

where

$$B_{q_n}(p_i r) = J_{q_n}(p_i r) \; Y_{q_n}(p_i) - Y_{q_n}(p_i r) \, J_{q_n}(p_i) \quad \ldots(13)$$

p_i is a positive root of the equation

$$J_{q_n}(p_i\sigma) \; Y_{q_n}(p_i) - Y_{q_n}(p_i\sigma) \, J_{q_n}(p_i) = 0 \quad \ldots(14)$$

and $J_{q_n}(p_i)$, $Y_{q_n}(p_i)$ are the Bessel functions of first and second kind of order q_n respectively, eqns. (3) and (4) transform using the boundary conditions (7) to (9) into

$$\frac{\partial \bar{u}_H}{\partial t} = \frac{(-1)^n}{q_n} \left\{\int_1^\sigma r \, . \, B_{q_n}(p_i r) \, dr\right\} f(t) - p_i^2 \, \bar{u}_H + (\bar{v}_H - \bar{u}_H) \quad \ldots(15)$$

and

$$\frac{\partial \bar{v}_H}{\partial t} = \gamma'(\bar{u}_H - \bar{v}_H) \quad \ldots(16)$$

where $-\partial\bar{p}/\partial\bar{z} = f(t)$, an arbitrary function of time t.

The initial conditions (5) are transformed to

$$\left.\begin{array}{l}\bar{u}_H(p_i, n, t) = 0 \\ \bar{v}_H(p_i, n, t) = 0\end{array}\right\} \text{ for } t \leqslant 0. \quad \ldots(17)$$

Now evaluating the integrals involving in eqn. (15) with the help of a known result Erdélyi (1953, p. 50), it can easily be shown that

$$\int_1^\sigma r \,.\, B_{q_n}(p_i r)\, dr = \frac{2}{\pi p_i^2}\left[\frac{J_{q_n}(p_i)}{J_{q_n}(p_i \sigma)}\, S_{1,q_n}(p_i \sigma) - S_{1,q_n}(p_i)\right], \qquad \ldots(18)$$

where $S_{1,q_n}(p_i \sigma)$ is the Lommel function.

Substitute the value of the integral from (18) in (15) to get

$$\frac{\partial \bar{u}_H}{\partial t} = \frac{(-1)^n}{q_n} \cdot \frac{2}{\pi p_i^2}\left[\frac{J_{q_n}(p_i)}{J_{q_n}(p_i \sigma)}\, S_{1,q_n}(p_i \sigma) - S_{1,q_n}(p_i)\right]$$

$$- p_i^2\, \bar{u}_H + (\bar{v}_H - \bar{u}_H). \qquad \ldots(19)$$

Now apply the Laplace transform with respect to t defined by

$$\bar{\bar{u}}_H(p_i, n, s) = \int_0^\infty \bar{u}_H(p_i, n, t)\, e^{-st}\, dt \qquad \ldots(20)$$

to eqns. (16) and (19) under the transformed initial conditions (17) and then solve the resulting equations for $\bar{\bar{u}}_H$ and $\bar{\bar{v}}_H$ to get

$$\bar{\bar{u}}_H = \frac{2(-1)^n}{\pi p_i^2\, q_n}\left[\frac{J_{q_n}(p_i)}{J_{q_n}(p_i \sigma)}\, S_{1,d_n}(p_i \sigma) - S_{1,q_n}(p_i)\right]$$

$$\times \left[\frac{(s + \gamma')\, \bar{f}(s)}{s^2 + (\beta + \gamma' + p_i^2)\, s + \gamma' p_i^2}\right] \qquad \ldots(21)$$

and

$$\bar{\bar{v}}_H = \frac{2(-1)^n}{\pi p_i^2 q_n}\left[\frac{J_{q_n}(p_i)}{J_{q_n}(p_i \sigma)}\, S_{1,q_n}(p_i \sigma) - S_{1,q_n}(p_i)\right]$$

$$\times \left[\frac{\bar{f}(s)}{s^2 + (\beta + \gamma' + p_i^2)\, s + \gamma' p_i^2}\right] \qquad \ldots(22)$$

where $\bar{f}(s)$ and $\bar{\bar{v}}_H$ are the Laplace transforms of $f(t)$ and $\bar{v}_H$ respectively.

Now to obtain u and v from these equations first invert the Laplace transform by convolution theorem and then apply inversion formulae for the consine and Hankel transform (Tranter 1968) to get

$$u(r, \theta, t) = \frac{2\pi}{\alpha} \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} \frac{(-1)^n J_{q_n}^2(p_i\sigma)\, B_{q_n}(p_i r) \cos q_n\theta}{q_n \{J_{q_n}^2(p_i) - J_{q_n}^2(p_i\sigma)\}(\beta_1 - \beta_2)}$$

$$\times \left[\frac{J_{q_n}(p_i)}{J_{q_n}(p_i\sigma)} S_{1,q_n}(p_i\sigma) - S_{1,q_n}(p_i)\right]$$

$$\times \left[\int_0^t f(t-x)\{(\beta_1 + \gamma') e^{\beta_1 x} - (\beta_2 + \gamma') e^{\beta_2 x}\}\, dx\right] \quad \ldots(23)$$

and

$$v(r, \theta, t) = \frac{2\pi}{\alpha} \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} \frac{(-1)^n \gamma' J_{q_n}^2(p_i\sigma)\, B_{q_n}(p_i r) \cos q_n\theta}{q_n \{J_{q_n}^2(p_i) - J_{q_n}^2(p_i\sigma)\}(\beta_1 - \beta_2)}$$

$$\times \left[\frac{J_{q_n}(p_i)}{J_{q_n}(p_i\sigma)} S_{1,q_n}(p_i\sigma) - S_{1,q_n}(p_i)\right]$$

$$\times \left[\int_0^t f(t-x)\{e^{\beta_1 x} - e^{\beta_2 x}\}\, dx\right] \quad \ldots(24)$$

where the summation for $i = 1$ to ∞ being over the positive roots of eqn. (14) and

$$\begin{matrix}\beta_1 \\ \beta_2\end{matrix} = -\tfrac{1}{2}\, [(\gamma' + \beta + p_i^2) \mp \{(\gamma' + \beta + p_i^2)^2 - 4\gamma' p_i^2\}^{1/2}]. \quad \ldots(25)$$

Equations (23) and (24) represent the most general solution of the problem considered, when flow takes place under the influence of an arbitrary function of time sufficiently well behaved for its Laplace transform to exist.

Particular Case : Constant Pressure Gradient

Substituting $f(t) = C$, where C is an absolute constant, in eqns. (23) and (24) on simplifying, we get

$$u(r, \theta, t) = \frac{2\pi C}{\alpha} \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} \frac{(-1)^n J_{q_n}^2(p_i\sigma)\, B_{q_n}(p_i r) \cos q_n\theta}{q_n p_i^2 \{J_{q_n}^2(p_i) - J_{q_n}^2(p_i\sigma)\}}$$

$$\times \left[\frac{J_{q_n}(p_i)}{J_{q_n}(p_i\sigma)} S_{1,q_n}(p_i\sigma) - S_{1,q_n}(p_i)\right]$$

$$\times \left[1 - \frac{(\beta_1 + p_i^2) e^{\beta_2 t} - (\beta_2 + p_i^2) e^{\beta_1 t}}{\beta_1 - \beta_2}\right] \quad \ldots(26)$$

and

$$v(r, \theta, t) = \frac{2\pi C}{\alpha} \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} \frac{(-1)^n J_{q_n}(p_i\sigma) B_{q_n}(p_i r) \cos q_n\theta}{q_n p_i^2 \{J^2_{q_n}(p_i) - J^2_{q_n}(p_i\sigma)\}}$$

$$\times \left[\frac{J_{q_n}(p_i)}{J_{q_n}(p_i\sigma)} S_{1,q_n}(p_i\sigma) - S_{1,q_n}(p_i)\right]\left[1 - \frac{\beta_1 e^{\beta_2 t} - \beta_2 e^{\beta_1 t}}{\beta_1 - \beta_2}\right]. \quad ...(27)$$

It can be easily shown that

$$\pi(4 - q_n^2) \sum_{i=1}^{\infty} \frac{J^2_{q_n}(p_i\sigma) B_{q_n}(p_i r)}{p_i^2 \{J^2_{q_n}(p_i) - J^2_{q_n}(p_i\sigma)\}}$$

$$\times \left[\frac{J_{q_n}(p_i)}{J_{q_n}(p_i\sigma)} S_{1,q_n}(P_i\sigma) - S_{1,q_n}(p\ )\right]$$

$$= \left[\frac{1}{\sigma^{q_n} - \sigma^{-q_n}} \{-(\sigma^{-q_n} - \sigma^2) r^{q_n} + (\sigma^{q_n} - \sigma^2) r^{-q_n}\} - r^2\right] \quad ...(28)$$

and

$$\tfrac{1}{4}\left(\frac{\cos 2\theta}{\cos 2\alpha} - 1\right) = \frac{2}{\alpha} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \cos q_n\theta}{q_n(4 - q_n^2)}. \quad ...(29)$$

In view of eqns. (28) – (29), eqns. (26) – (27) can be written as

$$u = \frac{C}{4}\left(\frac{\cos 2\theta}{\cos 2\alpha} - 1\right) r^2 - \frac{2C}{\alpha} \sum_{n=0}^{\infty} \left[\frac{(-1)^n \cos q_n\theta}{q_n(\sigma^{q_n} - \sigma^{-q_n})(4 - q_n^2)}\right.$$

$$\left.\times \{(\sigma^{-q_n} - \sigma^2) r^{q_n} - (\sigma^{q_n} - \sigma^2) r^{-q_n}\}\right]$$

$$- \frac{2\pi C}{\alpha} \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} \frac{(-1)^n J^2_{q_n}(p_i\sigma) B_{q_n}(p_i r) \cos q_n\theta}{q_n p_i^2 \{J^2_{q_n}(p_i) - J^2_{q_n}(p_i\sigma)\}}$$

$$\times \left[\frac{J_{q_n}(p_i)}{J_{q_n}(p_i\sigma)} S_{1,q_n}(p_i\sigma) - S_{1,q_n}(p_i)\right]$$

$$\times \left[\frac{(\beta_1 + p_i^2) e^{\beta_2 t} - (\beta_2 + p_i^2) e^{\beta_1 t}}{\beta_1 - \beta_2}\right] \quad ...(30)$$

and

$$v = \frac{C}{4}\left(\frac{\cos 2\theta}{\cos 2\alpha} - 1\right) r^2 - \frac{2C}{\alpha} \sum_{n=0}^{\infty} \left[\frac{(-1)^n \cos q_n\theta}{q_n(\sigma^{q_n} - \sigma^{-q_n})(4 - q_n^2)} \right.$$

$$\left. \times \{(\sigma^{-q_n} - \sigma^2)\, r^{q_n} - (\sigma^{q_n} - \sigma^2)\, r^{-q_n}\} \right]$$

$$- \frac{2\pi C}{\alpha} \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} \frac{(-1)^n J_{q_n}^2(p_i\sigma)\, B_{q_n}(p_i r) \cos q_n\theta}{q_n p_i^2 \{J_{q_n}^2(p_i) - J_{q_n}^2(p_i\sigma)\}}$$

$$\times \left[\frac{J_{q_n}(p_i)}{J_{q_n}(p_i\sigma)} S_{1,q_n}(p_i\sigma) - S_{1,q_n}(p_i) \right] \left[\frac{\beta_1 e^{\beta_2 t} - \beta_2 e^{\beta_1 t}}{\beta_1 - \beta_2} \right]. \qquad \ldots(31)$$

Equations (30) - (31) determine the velocities of the gas and particles respectively, under the influence of a constant pressure gradient C.

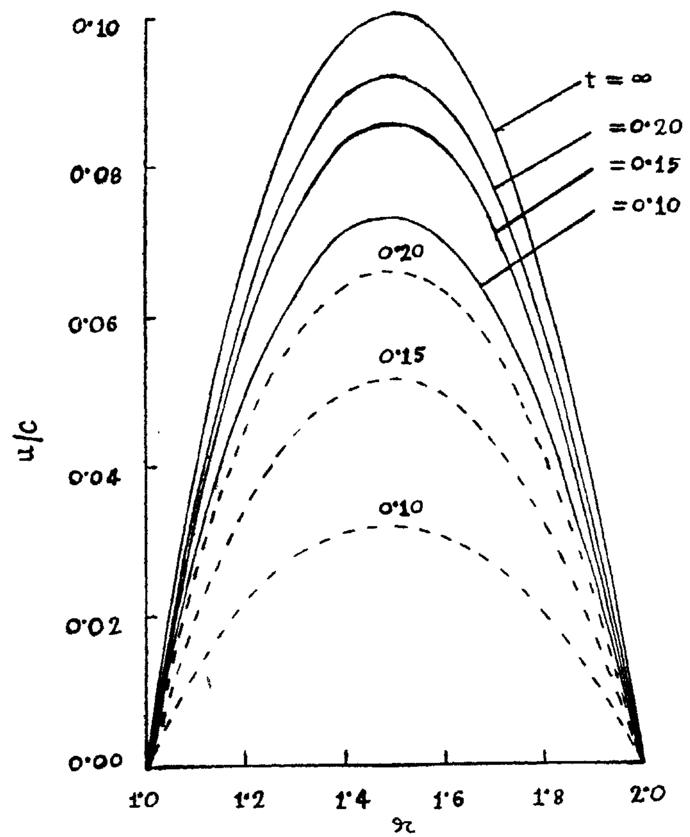

FIG. 2. Velocity profiles of gas and particle for values of t indicate when $\alpha = \pi/6$, $\theta = 0$ and $f = 0.1$, $\gamma = 0.1$.

Discussion

Equations (23) and (24) represent the velocities of the gas and particles for the general case. From these expressions various particular cases can easily be obtained for several values of $f(t)$. In Figs. 2 – 4, are plotted the velocity profiles for the gas and particles for a constant pressure gradient C. It is observed that (i) the gas moves faster than the particles and as time t increases since the start of the motion, the velocities approach their steady states, (ii) with the increase in θ, the velocities of the gas and particles decrease and (iii) the velocities of the gas and particles are maximum in the central plane of the channel.

Figure 5 shows some typical examples of the results obtained. It is observed that the gas-particle velocity difference is small when γ is small. For larger γ the gas-particle velocity difference is large. At low particle concentration the gas moves the particles with it at all times when γ is small. With larger γ, the gas flow is unaffected

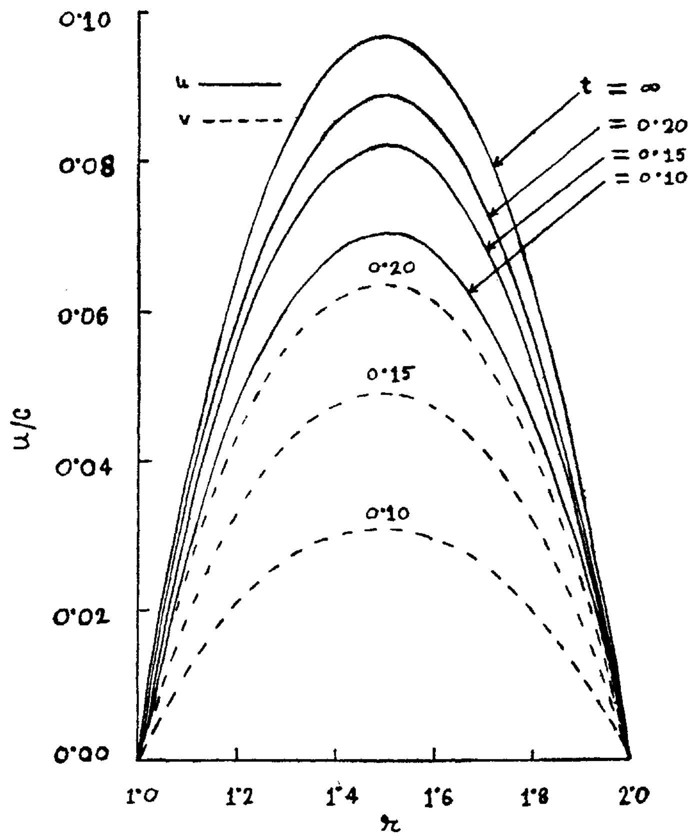

Fig. 3. Velocity profiles of gas and particle for values of t indicated when $\alpha = \pi/6$, $\theta = \pi/18$ and $f = 0.1$, $\gamma = 0.1$.

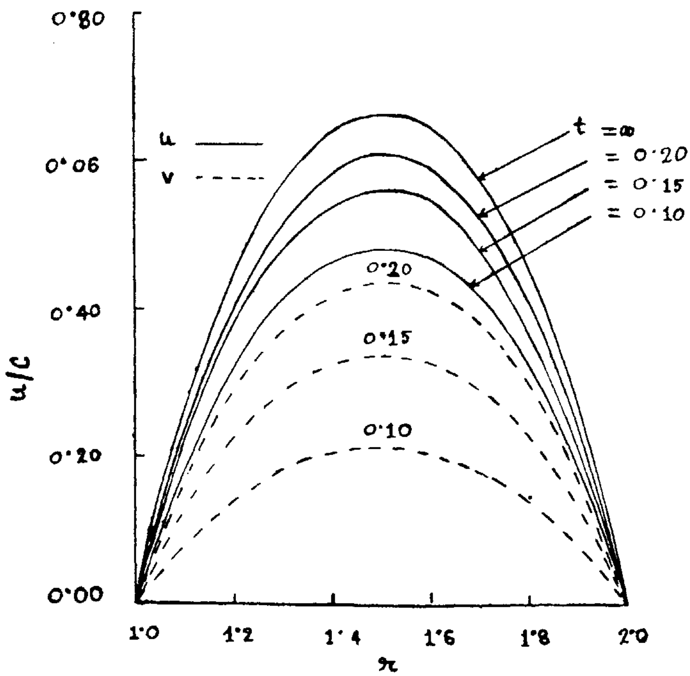

FIG. 4. Velocity profiles of gas and particle for values of t indicate when $\alpha = \pi/6$, $\theta = \pi/9$ and $f = 0.1$, $\gamma = 0.1$.

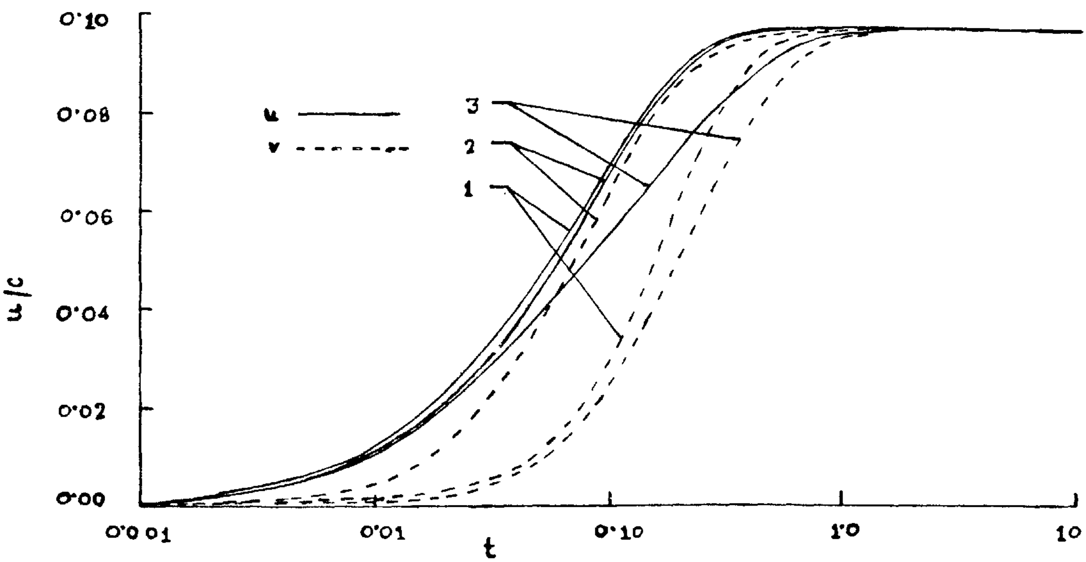

FIG. 5. Velocity profiles of gas and particle when $\alpha = \pi/6$, $\theta = \pi/18$ at $r = 1.5$ and (1) $f = 0.10$, $\gamma = 0.10$, (2) $f = 0.10$, $\gamma = 0.01$, (3) $f = 1.0$, $\gamma = 0.10$.

but the large gas-particle velocity results. When large particle concentrations are present lower particle and gas acceleration result. For larger γ, the gas starts accelerating sooner but the particles start later and the time to reach steady flow is also increased. Finally the gas and particle accelerate to the same final steady state value of the velocity.

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