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CONVERGING CYLINDRICAL DETONATION WAVES IN MAGNETOGASDYNAMICS

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The problem of cylindrical magnetogasdynamic detonation waves, convergent to the axis of symmetry through a gas with varying initial density, is studied. It is shown that the consideration of varying initial density affects the front velocity considerably. On the other hand, magnetic field has very small effect on the front velocity.

1. INTRODUCTION

The problem of imploding cylindrical or spherical shock front propagating into a uniform gas at rest is well known (Stanyukovich 1960). Nigmatulin (1967) has replaced the shock front by a contracting detonation front propagating into a uniform combustible gas. Here, we have considered the problem of converging cylindrical detonation waves in magnetogasdynamics. The electrical conductivity of the gas is supposed to be infinite and initial density of the gas varies as some power of distance measured from the axis of symmetry. The law of convergence is determined by the condition that the detonation wave is Chapman-Jouguet front i.e. it travels with velocity of propagation of small disturbances relative to the burnt gas (Helliwell 1963). The expression for similarity exponent is determined and graphs are plotted for velocity of detonation front, flow velocity, pressure, density and magnetic field. It is shown that when we take the initial density to be varying, the front velocity and flow parameters change considerably. The magnetic field, however, has very small effect on the front velocity as well as on the flow parameters.

2. BASIC EQUATIONS

Consider a cylindrically symmetric flow of a combustible gas with infinite electrical conductivity in which a strong detonation wave travels to the axis of symmetry. The origin of coordinates is taken on the axis of collapse. The time t is taken to be negative before the detonation front reaches the axis of symmetry and $t = 0$ is the instant of collapse. We assume that the undisturbed density ρ_0 and magnetic field h_0 ahead of the detonation wave are given respectively by

$$\rho_0 = A_0 r^{-k} \qquad \ldots(2.1a)$$

$$h_0 = B_0 r^{-k} \qquad \ldots(2.1b)$$

where A_0, B_0 and k are positive constants and r is the distance measured from the axis of symmetry. The position of the detonation front is assumed to be given by

$$r_2 = a_n(-t)^n \qquad \text{...(2.2)}$$

where a_n and $n(< 1)$ are constants. The basic equations governing the symmetric flow of a perfect, inviscid gas with infinite electrical conductivity in the Eulerian system are

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial r} + \rho\frac{\partial v}{\partial r} + \frac{\rho v}{r} = 0 \qquad \text{...(2.3)}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} + \frac{1}{\rho}\left(\frac{\partial p}{\partial r} + h\frac{\partial h}{\partial r}\right) = 0 \qquad \text{...(2.4)}$$

$$\frac{\partial h}{\partial t} + v\frac{\partial h}{\partial r} + h\frac{\partial v}{\partial r} + \frac{hv}{r} = 0 \qquad \text{...(2.5)}$$

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial r} + \gamma p\frac{\partial v}{\partial r} + \frac{\gamma p v}{r} = 0 \qquad \text{...(2.6)}$$

where v, ρ, p, h are velocity, density, pressure and magnetic field transverse to the flow, respectively, γ is the ratio of specific heats.

From the conditions for the conservations of mass, momentum, energy and magnetic field at the strong magnetogasdynamic detonation wave, we have

$$v_2 = \beta D \qquad \text{...(2.7}a\text{)}$$

$$p_2 = \beta\rho_1 D^2 \qquad \text{...(2.7}b\text{)}$$

$$\rho_2 = \rho_1/(1-\beta) \qquad \text{...(2.7}c\text{)}$$

$$h_2 = h_1/(1-\beta) \qquad \text{...(2.7}d\text{)}$$

where

$$\beta = \frac{1}{\gamma+1}\left\{1 + \left[1 - 2(\gamma-1)(\gamma+1)\frac{Q}{D^2}\right]^{1/2}\right\} = \frac{\alpha}{\gamma+1} \qquad \text{...(2.8)}$$

and D is the velocity of the detonation front. The suffixes 2 and 1 refer to conditions just behind and just ahead of the detonation front respectively, Q is the heat release per unit mass of the gas.

We introduce a dimensionless variable $\eta = r/r_2$ and seek a solution of the form

$$v = -\beta D V(\eta) \qquad \text{...(2.9}a\text{)}$$

$$p = \beta\rho_1 D^2 P(\eta) \qquad \text{...(2.9}b\text{)}$$

$$\rho = [\rho_1/(1-\beta)]\, G(\eta) \qquad \text{...(2.9}c\text{)}$$

$$h = [\beta\rho_1 D^2]^{1/2}\, H(\eta) \qquad \text{...(2.9}d\text{)}$$

At the detonation front, where $\eta = 1$, the boundary conditions (2.7) for the reduced functions $V(\eta)$, $P(\eta)$ and $H(\eta)$ become

$$V(1) = -1, \ P(1) = 1, \ G(1) = 1, \ H(1) = \frac{M_h^{-1}}{(1-\beta)\,\beta^{1/2}} \qquad \text{...(2.10)}$$

where M_h is the Alfvén Mach number of the detonation wave given by

$$M_h = \frac{D}{(h_1^2/\rho_1)^{1/2}}.$$

Converting eqns. (2.3) to (2.6) to the dimensionless variables in (2.9), we get the following system of four ordinary differential equations :

$$G'(\eta + \beta V) + \beta G V' + \beta G V/\eta + kG = 0 \qquad \text{...(2.11}a\text{)}$$

$$GV'(\eta + \beta V) + (1-\beta)\,[P' + HH'] - GV(n-1)/n = 0 \qquad \text{...(2.11}b\text{)}$$

$$H'(\eta + \beta V) + \beta H V' + \beta H V/\eta - \left(\frac{n-1}{n} - \frac{k}{2}\right) H = 0 \qquad \text{...(2.11}c\text{)}$$

$$P'(\eta + \beta V) + \gamma\beta P V' + \gamma\beta P V/\eta - 2\left(\frac{n-1}{n} - \frac{k}{2}\right) P = 0 \qquad \text{...(2.11}d\text{)}$$

where β is taken to be constant for the present but will be taken variable later. A quantity with dash represents the derivative of that quantity with respect to η. The determination of n will be examined in the next section.

3. The Convergence Rule

When the detonation front converges from infinity, the motion is defined by the parameters Q, A_0, r, t and γ. From dimensional considerations the similarity exponent n must be equal to 1, that is, the detonation front must travel with a constant velocity. But, a uniformly collapsing detonation cannot exist because as the detonation progresses its surface area diminishes, causing its velocity to increase towards the axis of symmetry, where it is infinite. Here, we do not intend to give a mathematical argument for the non-existence of a uniformly collapsing detonation since the mathematical argument can be found in Nigmatulin (1967) in the case when $k = 0$ and $M_h^{-1} = 0$.

In eqn. (2.8), $\alpha = 2$ corresponds to a shock wave without the heat release i.e. $Q = 0$; $\alpha = 1$ corresponds to the Chapman-Jouguet detonation regime, where the wave travels along characteristics in the disturbed gas.

We investigate the extreme case $\alpha = 1$ in the convergence process, which ensures the Chapman-Jouguet condition and the detonation wave travels along a characteristic. In this way we will seek the law for the convergence of the front as a characteristic dividing the disturbed and undisturbed media.

We use the equations of one-dimensional motion in Lagrangian variables

$$\rho_0 = (1 + u_r)\left(1 + \frac{u}{r}\right) \qquad \text{...(3.1a)}$$

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = -\left(1 + \frac{u}{r}\right)\frac{\partial}{\partial r}\left(p + \frac{h^2}{2}\right) \qquad \text{...(3.1b)}$$

$$\frac{p}{p_2} = \left(\frac{\rho}{\rho_2}\right)^{\gamma} = \left(\frac{h}{h_2}\right)^{\gamma} \qquad \text{...(3.1c)}$$

where $u_r \equiv \frac{\partial u}{\partial r}$ and u is a displacement of a particle. The conditions on the magneto-gasdynamic detonation wave in the Chapman-Jouguet regime are

$$v_2 = \frac{D}{\gamma + 1} \qquad \text{...(3.2a)}$$

$$p_2 = \frac{\rho_1 D^2}{\gamma + 1} \qquad \text{...(3.2b)}$$

$$\rho_2 = \frac{\rho_1(\gamma + 1)}{\gamma} \qquad \text{...(3.2c)}$$

$$h_2 = \frac{h_1(\gamma + 1)}{\gamma}. \qquad \text{...(3.2d)}$$

We now put eqn. (3.1*b*) in the form,

$$\frac{\partial^2 u}{\partial t^2} = a^2(r, u, u_r)\frac{\partial^2 u}{\partial r^2} + b(r, u, u_r) \qquad \text{...(3.3)}$$

so that the equations for the characteristics and the conditions on them are

$$dr = \pm\, a dt, \; du_t = \pm\, a du_r + b dt \qquad \text{...(3.4)}$$

where $u_t \equiv \frac{\partial u}{\partial t}$.

Using eqn. (3.1*a*) and relations (3.2) in eqn. (3.1*c*), we obtain,

$$\left(p + \frac{h^2}{2}\right) = \frac{1}{\gamma + 1}\left(\frac{\gamma}{\gamma + 1}\right)^{\gamma} \frac{\rho_1 D^2}{(1 + u_r)^{\gamma}\left(1 + \frac{u}{r}\right)^{\gamma}\left(\frac{r}{r_2}\right)^{k\gamma}}$$

$$+ \frac{h_1^2/2}{(1 + u_r)^2\left(1 + \frac{u}{r}\right)^2\left(\frac{r}{r_2}\right)^{2k}} \qquad \text{...(3.5)}$$

Differentiating eqn. (3.5) and substituting into eqn. (3.1*b*), we get

$$\frac{\partial^2 u}{\partial t^2} = \left(1 + \frac{u}{r}\right)\left(\frac{\rho_1}{\rho_0}\right)\left[L_1 \frac{\partial^2 u}{\partial r^2} + L_2\right] \qquad \text{...(3.6)}$$

where

$$L_1 = \left(\frac{\gamma}{\gamma + 1}\right)^{\gamma+1} \frac{D^2}{(1 + u_r)^{\gamma+1}\left(1 + \frac{u}{r}\right)^{\gamma}\left(\frac{r}{r_2}\right)^{k\gamma}} + \frac{h_1^2/\rho_1}{(1 + u_r)^3\left(1 + \frac{u}{r}\right)^2\left(\frac{r}{r_2}\right)^{2k}}$$

and

$$L_2 = \left(\frac{\gamma}{\gamma + 1}\right)^{\gamma+1} \frac{D^2}{(1 + u_r)^{\gamma}\left(1 + \frac{u}{r}\right)^{\gamma}\left(\frac{r}{r_2}\right)^{k\gamma}} \times \left\{\frac{(ru_r - u)}{\left(1 + \frac{u}{r}\right)r^2} + \frac{k}{r} - \frac{2}{\gamma D}\frac{dD}{dr}\right\} + \frac{h_1^2/\rho_1}{(1 + u_r)^2\left(1 + \frac{u}{r}\right)^2\left(\frac{r}{r_2}\right)^{2k}}\left\{\frac{(ru_r - u)}{\left(1 + \frac{u}{r}\right)r^2} + \frac{k}{r}\right\}.$$

Equating (3.6) and (3.3) we obtain

$$a^2(r, u, u_r) = \left(1 + \frac{u}{r}\right)\left(\frac{\rho_1}{\rho_0}\right)L_1 \qquad \text{...(3.7)}$$

$$b(r, u, u_r) = \left(1 + \frac{u}{r}\right)\left(\frac{\rho_1}{\rho_0}\right)L_2. \qquad \text{...(3.8)}$$

Since it has been assumed above that the detonation wave travels along a characteristic, we have

$$a_2 = a\,[r, 0, u_{r2}(r)] = D(r). \qquad \text{...(3.9)}$$

Hence eqn. (3.7) gives

$$\left(\frac{1}{\gamma + 1}\right)^{\gamma+1} \frac{1}{(1 + u_{r2})^{\gamma+1}} + \frac{M_h^{-2}}{(1 + u_{r2})^3} = 1. \qquad \text{...(3.10)}$$

Case I — If we neglect M_h^{-2},

$$u_{r2} = -\frac{1}{\gamma + 1}.$$

Case II — If M_h^{-2} is not neglected, then value of u_{r2} can be determined only when a value of γ is assigned. In the calculation that follows the numerical value of M_h is taken to be 10.

For $\gamma = 2$, that is, when $p = h^2 \left(\frac{p_2}{h_2^2}\right)$,

$$u_{r2} = -\ 0.3261.$$

For $\gamma = 3$, $u_{r2} = -\ 0.2455$.

Also, from eqn. (3.8), we get

$$b_2 = \left(\frac{\gamma}{\gamma + 1}\right)^{\gamma+1} \frac{D^2}{(1 + u_{r2})^\gamma} \left\{\frac{u_{r2}}{r} + \frac{k}{r} - \frac{2}{\gamma D}\frac{dD}{dr}\right\} + \frac{(h_1^2/\rho_1)}{(1 + u_{r2})^2} \left\{\frac{u_{r2}}{r} + \frac{k}{r}\right\}. \qquad ...(3.11)$$

For the characteristic which bounds the quiescent gas, we have

$$du = u_{t2}dt + u_{r2}dr = 0$$

or

$$u_{t2} \pm a_2 u_{r2} = 0. \qquad ...(3.12)$$

Substituting (3.12) into (3.4) and using the fact that u_{r2} is constant, we have

$$\mp\ u_{r2} da_2 = b_2 dt. \qquad ...(3.13)$$

As a consequence of eqns. (3.4), (3.9), (3.13) and (3.11), we obtain

$$|\ D\ | = \frac{A}{r_2^m} \qquad ...(3.14)$$

where

$$m = (0.2247\ -\ 0.6892k),\ \text{for}\ \gamma = 2$$

and

$$m = (0.2511 - 1.0240k),\ \text{for}\ \gamma = 3.$$

Integrating eqn. (3.14) and noting that the velocity, being directed towards the axis, is taken as negative, with the condition $r_2 = 0$ at $t = 0$, we have

$$r_2 = a_n(-\ t)^n,\ n = \frac{1}{m + 1}. \qquad ...(3.15)$$

Thus the law of the converging detonation front is obtained in the form which was assumed in section 2, and furthermore the similarity exponent n is determined in term of k, of course, for specific values of γ.

Having determined n, one can solve the system (2.11) numerically with the boundary conditions (2.10). Here, again, we do not intend to solve the system (2.11) but we are giving another approach for the solution to the present problem of converging detonation front.

4. Solutions

Nigmatulin (1967) has neglected the effect of variation of Q on the similarity exponent where $k = 0$, $M_h^{-1} = 0$. Here, in the present approach, we also, neglect the effect of variation of Q on the similarity exponent (convergence rule), that is, Q is taken to be constant in the convergence process.

We assume that at a certain initial radius R, the detonation front has initiated its own motion at the Chapman-Jouguet velocity ($\alpha = 1$),

$$D_0^2 = 2(\gamma - 1)(\gamma + 1) Q_0. \qquad \text{...(4.1)}$$

From the eqn. (3.14), we get the following relation for the motion of the front :

$$\frac{D}{D_0} = \left(\frac{R}{r_2}\right)^m. \qquad \text{...(4.2)}$$

Substituting the last expression into (2.8) and using eqns. (2.7), we obtain successively,

$$\alpha = 1 + \left[1 - \left(\frac{r_2}{R}\right)^{2m}\right]^{1/2} \qquad \text{...(4.3}a\text{)}$$

$$\frac{v_2}{v_{20}} = \alpha \left(\frac{R}{r_2}\right)^m \qquad \text{...(4.3}b\text{)}$$

$$\frac{p_2}{p_{20}} = \alpha \left(\frac{R}{r_2}\right)^{2m} \qquad \text{...(4.3}c\text{)}$$

$$\frac{\rho_2}{\rho_{20}} = \frac{\gamma}{\gamma - (\alpha - 1)} \qquad \text{...(4.3}d\text{)}$$

$$\frac{h_2}{h_{20}} = \frac{\gamma}{\gamma - (\alpha - 1)}. \qquad \text{...(4.3}e\text{)}$$

Here v_{20}, p_{20}, ρ_{20}, h_{20} are the values of v, p, ρ and h in the Chapman-Jouguet regime for the specified Q_0. Equations (4.2) – (4.3) give the solution in terms of the radius of the front.

5. Results and Discussion

For the velocity of detonation front to increase towards the axis of symmetry we must have from (3.14) m to be positive. Hence the range of k depends on γ and M_h^{-1} and for $\gamma = 3$ and $M_h^{-1} = 0.1$ it is $0 \leqslant k < 0.245$.

TABLE I

Values of n for different values of γ, k and M_h^{-1}

k	$\gamma = 2$		$\gamma = 3$	
	$M_h^{-1} = 0$	$M_h^{-1} = 0.1$	$M_h^{-1} = 0$	$M_h^{-1} = 0.1$
0	0.8182	0.8165	0.8000	0.7993
0.05	0.8411	0.8402	0.8333	0.8334
0.10	0.8654	0.8652	0.8696	0.8705
0.15	0.8911	0.8918	0.9091	0.9112
0.20	0.9182	0.9201	0.9524	0.9557

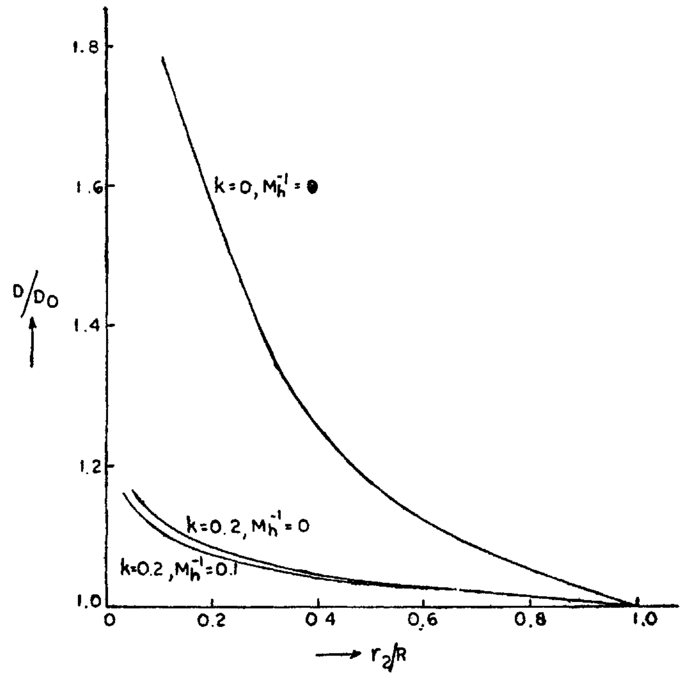


FIG. 1.

In Table I values of n are given for different values of γ, k and M_h^{-1}. It is found that n increases considerably due to increase in k. This increase, however, is less for $\gamma = 2$ than that for $\gamma = 3$. On the other hand, the effect of change of M_h^{-1} (i.e. the

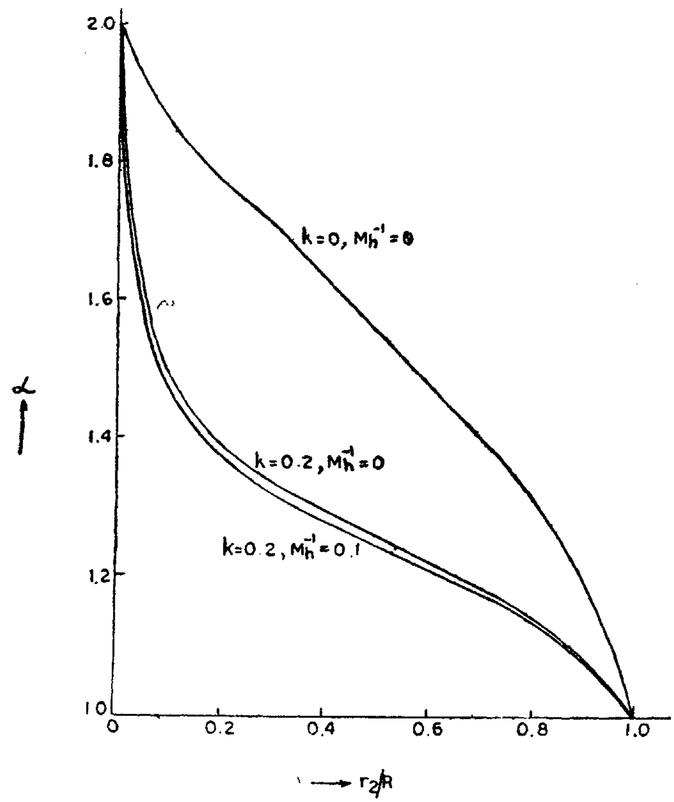
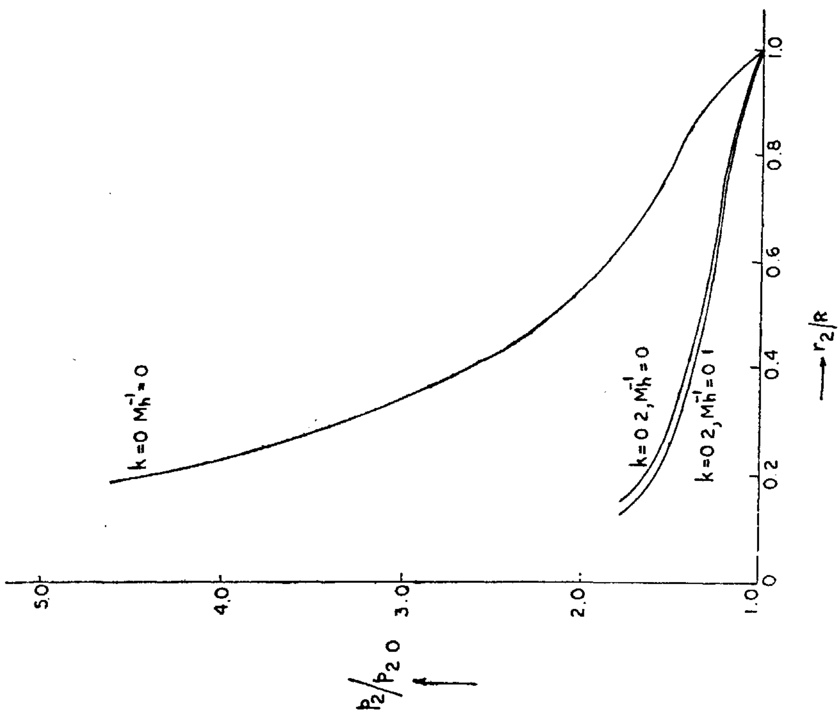


FIG. 2.

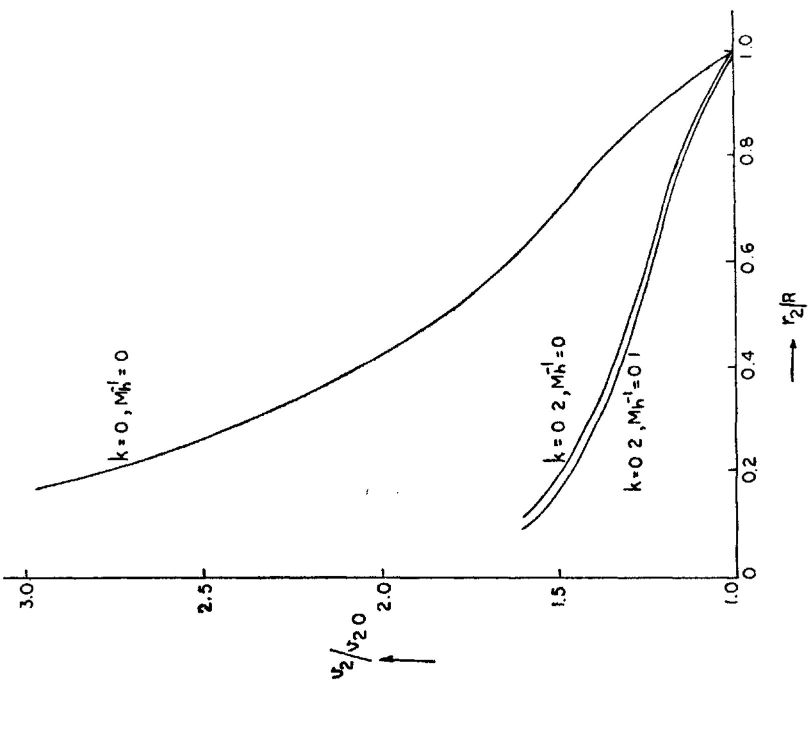
effect of magnetic field) on n is small. Also, for constant initial density ($k = 0$) effect of magnetic field is to decrease n (increase in front velocity) and for varying initial density ($k \neq 0$) its effect is to increase n (decrease in front velocity). We have also plotted the graphs (Figs. 1, 2, 3, 4 and 5) showing the variations of detonation front velocity, D/D_0, α, flow velocity v_2/v_{20}, pressure p_2/p_{20}, density ρ_2/ρ_{20} and magnetic field h_2/h_{20} in terms of front radius r_2/R for $\gamma = 3$, $k = 0, 0.2$ and $M_h^{-1} = 0, 0.1$. The case $k = 0$, $M_h^{-1} = 0$ corresponds to detonation waves converging to the axis of symmetry through a combustible gas of uniform initial density in ordinary gasdynamics (Nigmatulin 1967).

It is shown that, there is a remarkable change in the detonation front velocity, flow velocity, pressure and density, due to the consideration of varying initial density, except at initial point ($r_2 = R$) and end point ($r_2 = 0$). For $k = 0.2$, Fig. 1 indicates that the velocity of detonation front is almost constant, but near the



k=0 M_h^-1=0
k=0 2,M_h^-1=0
k=0 2,M_h^-1=0 1
p_2/p_20
r_2/R
5.0
4.0
3.0
2.0
1.0
0
0.2
0.4
0.6
0.8
1.0

FIG. 4.



k=0, M_h^-1=0
k=0 2, M_h^-1=0
k=0 2, M_h^-1=0 1
v_2/v_20
r_2/R
3.0
2.5
2.0
1.5
1.0
0
0.2
0.4
0.6
0.8
1.0

FIG. 3.

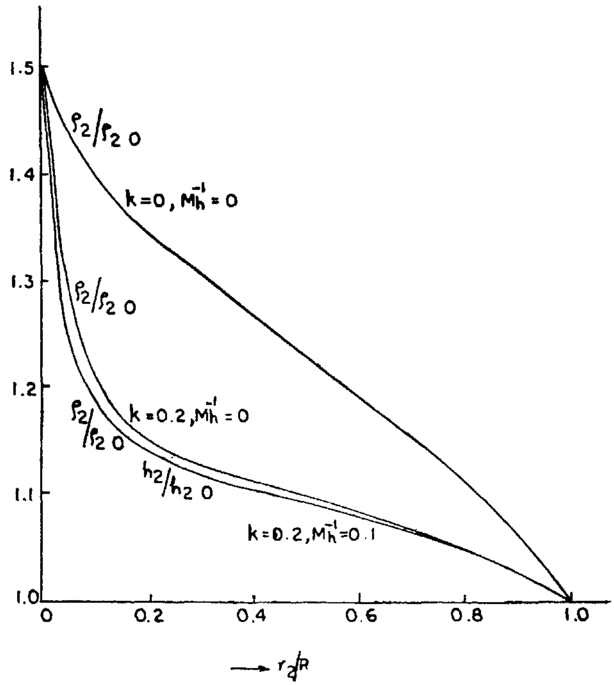


FIG. 5.

instant of collapse it begins to increase rapidly. The constancy of front velocity arises as a result of varying initial density. On the other hand, the effect of magnetic field is very small on the front velocity and flow parameters just behind the detonation wave. This is possible due to the fact that we have considered a strong detonation wave.

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