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SPECTRALLY CONVEX BANACH ALGEBRAS

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If A is a complex unital Banach algebra each element of which has a convex spectrum, then A modulo the radical of A is isomorphic to the complex numbers. The same conclusion is obtained for Banach star algebra A under the convexity hypothesis on the spectrum of each normal element only, and additionally, if A is also Hermitian, then it suffices to assume that the spectrum of each unitary element is convex.

Convex approximation of spectrum by numerical ranges in a unital complex Banach algebra (Bonsall and Duncan 1971, §1, Theorem 10, p. 23) suggests the problem of characterizing spectrally convex Banach algebras — Banach algebras A in which for each $x \in A$, its spectrum $\mathrm{sp}_A(x)$ is a convex set. This leads to certain characterizations of $\mathbb{C}$, the field of complex numbers among Banach algebras and Banach star algebras through spectral conditions.

First we prove the following.

Theorem 1 — If A is a spectrally convex complex unital Banach algebra, then $A/\mathrm{rad}\ A$ is isomorphic to $\mathbb{C}$.

Throughout, unless explicitly stated, A is a complex unital Banach algebra, the identity being denoted by 1. rad A denotes the Jacobson radical of A. For $x \in A$, $r(x)$ is the spectral radius of x and $N = \{x \in A \mid r(x) = 0\}$ is the set of all quasinilpotent elements of A. rad A is a closed two-sided ideal of A. For commutative Banach algebras A, rad $A = N$ and for each $x \in A$, $\mathrm{sp}_{A_1}(x + \mathrm{rad}\ A) = \mathrm{sp}_A(x)$, where $A_1 = A/\mathrm{rad}\ A$. In general, rad $A \subseteq N$. Recently, Zamanek (1977) have proved that in any Banach algebra A, the following are equivalent.

(i) rad $A = N$;

(ii) $x \in N$, $y \in N$ implies that $xy \in N$;

(iii) $x \in N$, $y \in N$ implies $x + y \in N$.

This leads us to the following lemma that we shall need for the proof of Theorem 1.

Lemma 2 — If $\mathrm{sp}_A(x)$ is a sigleton for each x in A, then rad $A = N$.

PROOF: We claim that for each a, b in A; $a \in N$, $b \in N$, implies that $ab \in N$. In fact, if a and b are in N, then $\mathrm{sp}_A(a) = \mathrm{sp}_A(b) = \{0\}$. If $\mathrm{sp}_A(ab) \neq \{0\}$, then $0 \notin \mathrm{sp}_A(ab)$ with the result ab is invertible in A and so a has a right inverse. Also, since by proposition 3 of Bonsall and Duncan (1971, §5, p. 20)

$$\mathrm{sp}_A(ab) \sim \{0\} = \mathrm{sp}_A(ba) \sim \{0\},\ 0 \notin \mathrm{sp}_A(ba)$$

which implies that a has a left inverse. Hence a is invertible contradicting the fact that $\mathrm{sp}_A(a) = \{0\}$.

Now by the above mentioned result of Zamanek (1977) the lemma follows.

For the proof of Theorem 1, we shall also need a part of numerical range theory in Banach algebras for which we refer to Bonsall and Duncan (1971). Recall (Bonsall and Duncan 1971, §13, Definition 1, p. 111) that a functional $f \in A'$, the dual of A, is a spectral state of A if $f(a) \in \mathrm{cosp}_A(a)$ for all $a \in A$ ($\mathrm{cosp}_A(a)$ is the convex hull of $\mathrm{sp}_A(a)$). $\Omega(A)$ denotes the set of all spectral states of A which is known to be non-empty (Bonsall and Duncan 1971, §13, p. 112) in case A is commutative.

Proof of Theorem 1 — Let A be spectrally convex, and for the moment assume that A is commutative and semi simple. By the well-known Gleason-Kahane-Zelazko theorem (Bonsall and Duncan 1973, §16, Theorem 7, p. 80), $\Omega(A) = \Phi_A$, the set of all complex homomorphisms of A. Since $\Omega(A)$ is convex, Φ_A is a convex subset of A', which in turn, as is easy to verify, forces Φ_A to be a singleton. Semi-simplicity of A now implies that

$$A \simeq \mathbb{C}. \qquad \text{...(1)}$$

In the general case, we assert that for each $x \in A$, $\mathrm{sp}_A(x)$ is a singleton. For, given $x \in A$, let M_x be the maximal commutative subalgebra of A containing x (Bonsall and Duncan 1971, §15, Theorem 4, p. 76). It is closed, cantains 1 and by Proposition 14 of Bonsall and Duncan (1973, p. 25) $\mathrm{sp}_A(y) = \mathrm{sp}_{M_x}(y)$ for each $y \in M_x$. This makes M_x and hence $M_x/\mathrm{rad}\ M_x$ spectrally convex. Since $M_x/\mathrm{rad}\ M_x$ is semi-simple, above (1) implies that $M_x/\mathrm{rad}\ M_x \simeq \mathbb{C}$. Therefore, for each $y \in M_x$, $\mathrm{sp}_{M_x}(y)$ is a singleton. In particular, $\mathrm{sp}_A(x)$ is a singleton, say $\{\beta_x\}$ for each $x \in A$.

Hence by above lemma, rad $A = N$. Now consider the semi-simple Banach algebra $A_1 = A/\mathrm{rad}\ A$. Since by Proposition 3 of (Bonsall and Duncan 1973, §5, p. 20), $\mathrm{sp}_{A_1}(x + \mathrm{rad}\ A) \subseteq \mathrm{sp}_A(x)$, $\mathrm{sp}_{A_1}(x + \mathrm{rad}\ A) = \{\beta_x\}$ for each $x \in A$. If, for any $x \in A$, $x \notin \mathrm{rad}\ A$, then $\beta_x \neq 0$ and so $0 \notin \mathrm{sp}_{A_1}(x + \mathrm{rad}\ A)$ with the result $x + \mathrm{rad}\ A$ is invertible in A_1. This shows that A_1 is a division algebra. Gelfand-Mazur theorem now completes the proof.

The following corollary is easy.

Corollary 3 — If there exists a neighbourhood V of 0 in A such that for each $x \in V$, $sp_A(x)$ is convex, then $A/\text{rad } A \simeq \mathbb{C}$.

It would be interesting to examine if we can replace, in above corollary, V by an arbitrary non-empty open set. We conjecture that the result still holds.

In passing we make the following remarks:

(a) The assertion of Theorem 1 does not hold for incomplete normed algebras. This is easily seen by considering the sup-norm algebra of all complex polynomials on the interval [0, 1] with the pointwise operations.

(b) As the example $C_{\mathbb{R}}[0, 1]$ of the sup-norm algebra of all continuous real valued functions on [0, 1] shows, the theorem fails for real Banach algebras.

(c) We do not know whether the spectral convexity of a non unital Banach algebra characterizes radical algebras.

Now we consider the modifications of Theorem 1 to Banach star algebras. For Banach star algebras we refer to Bonsall and Duncan (1973). Throughout the remainder of the paper, A denotes a complex unital Banach star algebra with 1. Recall that an element $x \in A$ is called normal if $xx^* = x^*x$. By Proposition 7 of Bonsall and Duncan 1973, §36, p. 190) each normal element x of A is contained in a maximal commutative * subalgebra B_x of A which is a closed subalgebra of A containing 1. $\text{Sym } A = \{x \in A \mid x = x^*\}$ is the set of all self-adjoint elements of A. The star-radical of A (Bonsall and Duncan 1973, §40, Definition 7, p. 223), denoted by srad A is the intersection of the kernels of all irreducible * representations of A on Hilbert space; and rad $A \subseteq$ srad A. Also, A is called Hermitian if for each $h \in \text{Sym } A$, $sp_A(h) \subset \mathbb{R}$, and for Hermitian Banach star algebras, by (Bonsall and Duncan 1973, §41, Theorem 9, p. 227), rad A = srad A.

Our desired modification of Theorem 1 is as follows.

Theorem 4 — If each normal element of A has a convex spectrum, then $A/\text{rad } A \simeq \mathbb{C}$.

First we mention a lemma. We say that A has locally a property P if for each $h \in \text{Sym } A$, the closed * subalgebra C_h, generated by 1 and h has the property P. Recently Cuntz (1976, Th. 1) has proved that if A is locally B^*-equivalent (i.e. for each $h \in \text{Sym } A$, C_h is topologically * isomorphic to a B^*-algebra), then A is B^*-equivalent (i.e. it is itself topologically * isomorphic to a B^*-algebra). The following is an easy consequence of this.

Lemma 5 — (a) If A is locally finite dimensional, then A is finite dimensional.

(b) If A is locally isomorphic to $\mathbb{C}$, then A is isomorphic to $\mathbb{C}$.

PROOF : (a) In case A is locally finite dimensional, then A is locally B^*-equivalent. Hence by above mentioned result of Cuntz (1976) A is a B^*-equivalent algebra, each self-adjoint element of which has a finite spectrum. Hence by remark after Theorem 1 of Aupetit (1976), A is finite dimensional.

(b) By (a), A is a finite dimensional B^*-algebra each self-adjoint element of which has a singleton spectrum. Hence by Proposition 20 (§12) Lemma 3 (§38) and Proposition 4 (§10) of Bonsall and Duncan (1973), the numerical range of each element of A is a singleton. Corollary 15 of Bonsall and Duncan (1973, p. 56, §10) now implies that A is $\mathbb{C}$.

Proof of Theorem 4 — For each normal element x of A, consider B_x. Since each $y \in B_x$ is obviously normal, the hypothesis of the theorem together with Proposition 14 of Bonsall and Duncan (1973, p. 25, §5) implies that B_x is spectrally convex. Hence by Theorem 1, $B_x/\text{rad } B_x \simeq \mathbb{C}$ which shows that $\text{sp}_A(x)$ is a singleton for each normal x in A. In particular, for each $h \in \text{Sym } A$, $\text{sp}_A(h)$ is a singleton. This by Lemma 10 of Bonsall and Duncan (1973, §12, p. 65) implies that A is Hermitian and so rad A = srad A.

Now consider the semi-simple Banach star algebra $A_1 = A/\text{rad } A$, with the canonical involution which it admits from that of A. Given $h \in \text{Sym } A_1$, it is now easy to see that $\text{sp}_{C_h}(x)$ is a singleton for each $x \in C_h$, C_h denoting the closed subalgebra of A_1 generated by 1 and h. Since A_1 is also Hermitian, it is an A^*-algebra by Corollary 10 of Bonsall and Duncan (1973, p. 227) and so is C_h for each $h \in \text{Sym } A_1$. In particular, C_h is semi-simple for $h \in \text{Sym } A_1$. Since a semi-simple commutative Banach algebra with identity each element of which has a singleton spectrum is easily verified to be isomorphic to $\mathbb{C}$, (or even Theorem 1), it follows that A_1 is locally isomorphic to $\mathbb{C}$. Hence by Lemma 5, A_1 is isomorphic to $\mathbb{C}$. This completes the proof of the theorem.

The tempting conjecture that a semi-simple Banach star algebra each self-adjoint element of which has a convex spectrum should be $\mathbb{C}$ is obviously false as the consideration of the B^*-algebra $C[0, 1]$ of all continuous complex valued functions on $[0, 1]$ shows. However, implicitly contained in the proof of above theorem is the following theorem.

Theorem 6 — If A is a Hermitian Banach star algebra each unitary element of which has a convex spectrum, then $A/\text{rad } A$ is isomorphic to $\mathbb{C}$.

Recall that an element x of A is unitary if $xx^* = x^*x = 1$. If $\mathcal{U}$ denotes the set of all unitary elements of A, then by Proposition 14 of Bonsall and Duncan (1971, p. 66), A is a linear span of $\mathcal{U}$. In fact, as in the course of the proof there, given $h \in \text{Sym } A$ with $r(h) < 1$, there exists $u \in \mathcal{U}$ such that $h = \frac{1}{2}(u + u^*)$. The following is an obvious generalization of the well-known B^*-algebra result.

Lemma 7 — The spectrum of a unitary element in a Hermitian Banach star algebra is contained in the unit circle.

PROOF : For $u \in \mathcal{U}$, if B_u is the maximal commutative *-subalgebra containing u, then by Proposition 14 of Bonsall and Duncan (1973, p. 25),

$$\mathrm{sp}_A(u) = \mathrm{sp}_{B_u}(u) = \{\varphi(u) \mid \varphi \in \Phi_{B_u}\} \text{ and } \mathrm{sp}_{B_u}(y) = \mathrm{sp}_A(y)$$

for each $y \in \mathrm{Sym}\, B_u$. Hence B_u is Hermitian. Theorem 3 of Bonsall and Duncan (1973, p. 188) now implies that for all $\varphi \in \Phi_{B_u}$,

$$1 = \varphi(uu^*) = \varphi(u)\,\varphi(u^*) = \varphi(u)\,\overline{\varphi(u)} = |\,\varphi(u)\,|^2.$$

Hence the lemma.

Proof of Theorem 6 — The hypothesis of theorem together with above lemma implies that for each $u \in \mathcal{U}$, $\mathrm{sp}_A(u)$ is a singleton. Remark preceding the statement of Lemma 7 shows that the spectrum of each self-adjoint element of A is singleton. The rest of the proof now follows the same pattern as that of Theorem 4.

The following result of Fields (1972) follows as a corollary of Theorem 6.

Corollary 8 — A unital B^*-algebra each unitary element of which has a convex spectrum is isomorphic to $\mathbb{C}$.

We do not know whether the hypothesis of Hermiticity can be omitted from Theorem 6.

Finally we may remark that the results of the paper can be suitably generalized to locally bounded and locally convex algebras. We hope to discuss them in a subsequent paper.

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REFERENCES

Aupetit, B. (1976). On scarcity of operators with finite spectrum. *Bull. Am. math. Soc.*, **82** (3), 485-86.

Bonsall, F. P., and Duncan, J. (1971). *Numerical ranges of operators on normed spaces and of elements of normed algebras. London Math. Soc. Lecture Note Series* 2. Cambridge University Press, Cambridge.

——————— (1973). Complete Normed Algebras. Springer-Verlag, Berlin.

Cuntz, J. (1976). Locally C^*-equivalent algebras. *J. Functional Analysis*, **23** (2), 95-107.

Fields, J. W. (1972). Linear mappings of Banach algebra and Banach algebras of vector-valued functions. Thesis, Texas Christian University.

Zamanek, J. (1977). A note on the radical of a Banach algebra. *Manuscripta Math.*, **20**, 191-96.