Indian J. pure appl. Math., 10(1) : 74-83, January 1979

ON KAEHLERIAN CONCIRCULAR RECURRENT AND KAEHLERIAN CONCIRCULAR SYMMETRIC SPACES

by A. K. SINGH*, *Department of Mathematics, University of Gorakhpur, Gorakhpur* 273001

(*Received* 15 *May* 1978)

Sinha (1973) has studied *H*-curvature tensors in Kaehler manifold. In the present paper, we have defined and studied Kaehlerian concircular recurrent and Kaehlerian concircular symmetric spaces. The necessary and sufficient condition that a Kaehler space be Kaehlerian concircular recurrent has been established. Several other theorems have also been investigated. Further, we have defined Kaehlerian concircular symmetric space and have shown that the scalar curvature R is constant in a Kaehlerian concircular symmetric space. The necessary and sufficient condition that a Kaehlerian concircular symmetric space be Kaehlerian Ricci-recurrent with the same recurrence vector has been derived.

1. INTRODUCTION

An $n(=2m)$ dimensional Kaehlerian space K^c_n is a Riemannian space which admits a structure tensor field F^h_i satisfying the relations (Yano 1965)

$$F^i_j F^h_i = -\delta^h_j \qquad \ldots(1.1)\dagger$$

$$F_{ij} = -F_{ji}, \quad (F_{ij} = F^a_i \, g_{aj}) \qquad \ldots(1.2)$$

and

$$F^h_{i,j} = 0 \qquad \ldots(1.3)$$

where the comma followed by an index denotes the operation of covariant differentiation with respect to the metric tensor g_{ij} of the Riemannian space.

The Riemannian curvature tensor which we denote by R^h_{ijk} is given by

$$R^h_{ijk} = \partial_i \begin{Bmatrix} h \\ jk \end{Bmatrix} - \partial_j \begin{Bmatrix} h \\ ik \end{Bmatrix} + \begin{Bmatrix} h \\ ia \end{Bmatrix} \begin{Bmatrix} a \\ jk \end{Bmatrix} - \begin{Bmatrix} h \\ ja \end{Bmatrix} \begin{Bmatrix} a \\ ik \end{Bmatrix}$$

*Present address : Department of Mathematics, Garhwal University College, Pauri Garhwal 246001.

†All the Latin indices run over the same range from 1 to n.

whereas the Ricci-tensor and the scalar curvature are respectively given by $R_{ij} = R^a_{aij}$ and $R = R_{ij}g^{ij}$.

It is well known that these tensors satisfy the identities (Tachibana 1967)

$$F^a_i R^j_a = R^a_i F^j_a \qquad \ldots(1.4)$$

and

$$F^a_i R_{aj} = - R_{ia} F^a_j . \qquad \ldots(1.4a)$$

In view of (1.1), the relation (1.4) gives

$$F^a_i R^b_a F^j_b = - R^j_i . \qquad \ldots(1.5)$$

Also, multiplying (1.4*a*) by g^{ij}, we obtain

$$F^a_i R^i_a = - R^j_a F^a_j$$

which implies

$$F^a_i R^i_a = 0. \qquad \ldots(1.6)$$

If we define a tensor S_{ij} by

$$S_{ij} = F^a_i R_{aj} \qquad \ldots(1.7)$$

we have

$$S_{ij} = - S_{ji}. \qquad \ldots(1.8)$$

The holomorphically concircular curvature tensor C^h_{ijk} and the Bochner curvature tensor K^h_{ijk} are respectively given by (Sinha 1973)

$$C^h_{ijk} = R^h_{ijk} + \frac{R}{n(n+2)} (g_{ik}\, \delta^h_j - g_{jk}\, \delta^h_i + F_{ik} F^h_j - F_{jk} F^h_i + 2F_{ij} F^h_k) \qquad \ldots(1.9)$$

$$K^h_{ijk} = R^h_{ijk} + \frac{1}{n+4} (R_{ik}\, \delta^h_j - R_{jk}\, \delta^h_i + g_{ik} R^h_j - g_{jk} R^h_i + S_{ik} F^h_j - S_{jk} F^h_i + F_{ik} S^h_j - F_{jk} S^h_i + 2S_{ij} F^h_k + 2F_{ij} S^h_k) - \frac{R}{(n+2)(n+4)} \times (g_{ik}\, \delta^h_j - g_{jk}\delta^h_i + F_{ik} F^h_j - F_{jk} F^h_i + 2F_{ij} F^h_k). \qquad \ldots(1.10)$$

Equation (1.10), in view of (1.9) will be expressed in the form

$$K^{h}_{ijk} = C^{h}_{ijk} + \frac{1}{n+4}(R_{ik}\delta^{h}_{j} - R_{jk}\delta^{h}_{i} + g_{ik}R^{h}_{j} - g_{jk}R^{h}_{i} + S_{ik}F^{h}_{j} - S_{jk}F^{h}_{i} + F_{ik}S^{h}_{j} - F_{jk}S^{h}_{i} + 2S_{ij}F^{h}_{k} + 2F_{ij}S^{h}_{k}) - \frac{2R}{n(n+4)}(g_{ik}\delta^{h}_{j} - g_{jk}\delta^{h}_{i} + F_{ik}F^{h}_{j} - F_{jk}F^{h}_{i} + 2F_{ij}F^{h}_{k}). \quad ...(1.11)$$

We shall use the following definitions :

Definition 1.1 — A Kaehler space K^{c}_{n} satisfying (Lal and Singh 1971)

$$R^{h}_{ijk,a} - \lambda_a R^{h}_{ijk} = 0 \quad ...(1.12)$$

for some non-zero vector field λ_a will be called a Kaehlerian recurrent space. The space K^{c}_{n} is called Kaehlerian Ricci-recurrent if it satisfies the relation

$$R_{ij,a} - \lambda_a R_{ij} = 0. \quad ...(1.13)$$

Multiplying the above equation by g^{ij} and using the fact that $g^{ij}_{,a} = 0$, we get

$$R_{,a} - \lambda_a R = 0. \quad ...(1.14)$$

Remark 1.1 : From (1.13), it follows that every Kaehlerian recurrent space is Kaehlerian Ricci-recurrent but the converse is not necessarily true.

Definition 1.2 — The space K^{c}_{n} is called Kaehlerian symmetric in the sense of Cartan if it satisfies (Lal and Singh 1971)

$$R^{h}_{ijk,a} = 0, \text{ or equivalently } R_{ijkl,a} = 0. \quad ...(1.15)$$

Obviously, a Kaehlerian symmetric space is Kaehlerian Ricci-symmetric, i.e.

$$R_{ij,a} = 0. \quad ...(1.16)$$

Definition 1.3 — The space K^{c}_{n} in which the Bochner (*H*-conformal) curvature tensor K^{h}_{ijk} satisfies the relation (Lal and Singh 1971)

$$K^{h}_{ijk,a} - \lambda_a K^{h}_{ijk} = 0 \quad ...(1.17)$$

for some non-zero vector field λ_a, will be called a Kaehler space with recurrent Bochner curvature tensor or Kaehlerian-Bochner recurrent space.

2. Kaehlerian Concircular Recurrent Space

Definition 2.1 — The space K_n^c satisfying the relation

$$C^h_{ijk,a} - \lambda_a C^h_{ijk} = 0 \qquad \text{...(2.1)}$$

for some non-zero recurrence vector field λ_a, will be called a Kaehlerian concircular recurrent space.

Theorem 2.1 — Every Kaehlerian recurrent space is Kaehlerian concircular recurrent.

Proof : A Kaehlerian recurrent space is characterized by the eqn. (1.12), which yields (1.14).

Differentiating (1.9) covariantly with respect to x^a, we obtain

$$C^h_{ijk,a} = R^h_{ijk,a} + \frac{R_{,a}}{n(n+2)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k). \qquad \text{...(2.2)}$$

Multiplying (1.9) by λ_a and subtracting from (2.2), we obtain

$$C^h_{ijk,a} - \lambda_a C^h_{ijk} = R^h_{ijk,a}\ \lambda_a\ R^h_{ijk} + \frac{(R_{,a} - \lambda_a R)}{n(n+2)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k). \qquad \text{...(2.3)}$$

Making use of the eqns. (1.12) and (1.14), we have

$$C^h_{ijk,a} - \lambda_a C^h_{ijk} = 0$$

which shows that the space is Kaehlerian concircular recurrent.

Theorem 2.2 — The necessary and sufficient condition that a space K_n^c is Kaehlerian Ricci-recurrent is that

$$C^h_{ijk,a} - \lambda_a C^h_{ijk} = R^h_{ijk,a} - \lambda_a R^h_{ijk}.$$

Proof : Let the space be Kaehlerian Ricci-recurrent, then the relation (1.13) is satisfied which yields (1.14). The eqn. (2.3), in view of (1.14), reduces to

$$C^h_{ijk,a} - \lambda_a C^h_{ijk} = R^h_{ijk,a} - \lambda_a R^h_{ijk}. \qquad \text{...(2.4)}$$

Conversely, if in a space K_n^c, eqn. (2.4) is satisfied, then we have from (2.3), the relation

$$\frac{(R_{,a} - \lambda_a R)}{n(n+2)}(g_{ik}\delta_j^h - g_{jk}\delta_i^h + F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h) = 0 \qquad \text{...(2.5)}$$

which yields

$$R_{ij,a} - \lambda_a R_{ij} = 0$$

i.e. the space is Kaehlerian Ricci-recurrent.

Theorem 2.3 — Every Kaehlerian concircular recurrent space is a Kaehler space with recurrent Bochner curvature tensor.

PROOF : Let the space be Kaehlerian concircular recurrent.

Equation (2.1), in view of (1.9), gives

$$R^h_{ijk,a} + \frac{R_{,a}}{n(n+2)}(g_{ik}\delta_j^h - g_{jk}\delta_i^h + F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h) - \lambda_a\left\{R^h_{ijk} + \frac{R}{n(n+2)}(g_{ik}\delta_j^h - g_{jk}\delta_i^h + F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h)\right\} = 0 \qquad \text{...(2.6)}$$

or

$$R^h_{ijk,a} - \lambda_a R^h_{ijk} + \frac{(R_{,a} - \lambda_a R)}{n(n+2)}(g_{ik}\delta_j^h - g_{jk}\delta_i^h + F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h) = 0. \qquad \text{...(2.6a)}$$

Contracting indices h and k in the above equation, we have

$$R_{ij,a} - \lambda_a R_{ij} + \frac{(R_{,a} - \lambda_a R)}{n(n+2)}(F_{ih}F_j^h - F_{jh}F_i^h + 2F_{ij}F_h^h) = 0. \qquad \text{...(2.7)}$$

Since $R = R_{ij}g^{ij}$, the above equation may be written as

$$(R_{ij,a} - \lambda_a R_{ij}) + \frac{(R_{ij,a} - \lambda_a R_{ij})}{n(n+2)}\{g^{ij}(F_{ih}F_j^h - F_{jh}F_i^h + 2F_{ij}F_h^h)\} = 0 \qquad \text{...(2.8)}$$

or

$$(R_{ij,a} - \lambda_a R_{ij}) \left\{ 1 - \frac{1}{n(n+2)} g^{ij} (F_{ih} F^h_j - F_{jh} F^h_i + 2F_{ij} F^h_h) \right\} = 0 \qquad \ldots(2.8a)$$

which implies

$$R_{ij,a} - \lambda_a R_{ij} = 0 \qquad \ldots(2.9)$$

i.e. the space is Kaehlerian Ricci-recurrent.

Multiplying (2.9) by g^{ij} and using the fact that $g^{ij}_{,a} = 0$, we obtain

$$R_{,a} - \lambda_a R = 0. \qquad \ldots(2.10)$$

Differentiating (1.11) covariantly with respect to x^a, we have

$$\begin{aligned} K^h_{ijk,a} = C^h_{ijk,a} &+ \frac{1}{n+4}(\delta^h_j R_{ik,a} - \delta^h_i R_{jk,a} \\ &+ g_{ik} R^h_{j,a} - g_{jk} R^h_{i,a} + F^h_j S_{ik,a} - F^h_i S_{jk,a} \\ &+ F_{ik} S^h_{j,a} - F_{jk} S^h_{i,a} + 2F^h_k S_{ij,a} + 2F_{ij} S^h_{k,a}) \\ &- \frac{2R_{,a}}{n(n+4)} (g_{ik} \delta^h_j - g_{jk} \delta^h_i + F_{ik} F^h_j \\ &- F_{jk} F^h_i + 2F_{ij} F^h_k). \end{aligned} \qquad \ldots(2.11)$$

Multiplying (1.11) by λ_a and subtracting from (2.11), we get

$$\begin{aligned} K^h_{ijk,a} - \lambda_a K^h_{ijk} = C^h_{ijk,a} - \lambda_a C^h_{ijk} & \\ + \frac{1}{n+4} \Big\{ \delta^h_j (R_{ik,a} - \lambda_a R_{ik}) & \\ - \delta^h_i (R_{jk,a} - \lambda_a R_{jk}) + g_{ik} (R^h_{j,a} & \\ - \lambda_a R^h_j) - g_{jk} (R^h_{i,a} - \lambda_a R^h_i) & \\ + F^h_j (S_{ik,a} - \lambda_a S_{ik}) - F^h_i (S_{jk,a} & \\ - \lambda_a S_{jk}) + F_{ik} (S^h_{j,a} - \lambda_a S^h_j) & \\ - F_{jk} (S^h_{i,a} - \lambda_a S^h_i) + 2F^h_k (S_{ij,a} - \lambda_a S_{ij}) + & \end{aligned}$$

(*equation continued on p.* 80)

$$+ 2F_{ij} (S^h_{k,a} - \lambda_a S^h_k) - \frac{2(R_{,a} - \lambda_a R)}{n(n+4)}$$

$$\times (g_{ik} \delta^h_j - g_{jk} \delta^h_i + F_{ik} F^h_j - F_{jk} F^h_i + 2F_{ij} F^h_k). \qquad \ldots(2.12)$$

Making use of eqns. (1.7), (2.1), (2.9) and (2.10) in (2.12), we get

$$K^h_{ijk,a} - \lambda_a K^h_{ijk} = 0,$$

which shows that the space is a Kaehler space with recurrent Bochner curvature tensor.

Theorem 2.4 — The necessary and sufficient conditions for a Kaehler space to be Kaehlerian concircular recurrent are that the space is a Kaehlerian Ricci-recurrent and a Kaehlerian-Bochner recurrent both.

PROOF : The necessary part has been proved in Theorem 2.3.

For the sufficient part, let us suppose that the space be both Kaehlerian Ricci-recurrent and Kaehlerian-Bochner recurrent. Then eqns. (1.13), (1.14) and (1.17) are satisfied.

The eqn. (1.11) yields (2.12), which in view of eqns. (1.13), (1.14) and (1.17), reduces to

$$C^h_{ijk,a} - \lambda_a C^h_{ijk} = 0.$$

This shows that the space is Kaehlerian concircular recurrent. Hence the sufficient part is proved.

This completes the proof.

Theorem 2.5 — A Kaehlerian concircular recurrent space will be Kaehlerian recurrent provided that it is Kaehlerian Ricci-recurrent.

PROOF : With the help of eqns. (1.9) and (2.1), we obtain (2.3), i.e.

$$C^h_{ijk,a} - \lambda_a C^h_{ijk} = R^h_{ijk,a} - \lambda_a R^h_{ijk}$$

$$+ \frac{(R_{,a} - \lambda_a R)}{n(n+2)} (g_{ik} \delta^h_j - g_{jk} \delta^h_i$$

$$+ F_{ik} F^h_j - F_{jk} F^h_i + 2F_{ij} F^h_k).$$

Now, let the space be Kaehlerian Ricci-recurrent. Therefore, eqns. (1.13) and (1.14) are satisfied. Making use of (1.14), the above equation reduces to

$$C^h_{ijk,a} - \lambda_a C^h_{ijk} = R^h_{ijk,a} - \lambda_a R^h_{ijk}.$$

This shows that the Kaehlerian concircular recurrent space is Kaehlerian recurrent.

Theorem 2.6 — If a Kaehler space satisfies any two of the properties :

(i) the space is Kaehlerian recurrent,

(ii) the space is Kaehlerian Ricci-recurrent,

(iii) the space is Kaehlerian concircular recurrent,

it must also satisfy the third.

PROOF : Kaehlerian recurrent, Kaehlerian Ricci-recurrent and Kaehlerian concircular recurrent spaces are respectively characterized by eqns. (1.12), (1.13) and (2.1). The statement of the theorem follows in view of eqns. (1.12), (1.13), (2.1) and (2.3).

3. KAEHLERIAN CONCIRCULAR SYMMETRIC SPACE

Definition 3.1 — A Kaehler space satisfying the relation

$$C^h_{ijk,a} = 0 \qquad \text{...(3.1)}$$

will be called a Kaehlerian concircular symmetric space.

Theorem 3.1 — Every Kaehlerian symmetric space is Kaehlerian concircular symmetric space.

PROOF : If the space is Kaehlerian symmetric then the relations (1.15) and (1.16) hold.

Differentiating (1.9) covariantly with respect to x^a and using (1.15) and (1.16), we get

$$C^h_{ijk,a} = 0.$$

This shows that the space is Kaehlerian concircular symmetric.

Theorem 3.2 — The necessary and sufficient condition that a Kaehlerian concircular symmetric space is Kaehlerian Ricci-recurrent with the same recurrence vector field λ_a is that

$$R^h_{ijk,a} + \lambda_a (C^h_{ijk} - R^h_{ijk}) = 0.$$

PROOF : With the help of eqns. (1.9) and (2.1), we obtain (2.3).

Since the space is Kaehlerian concircular symmetric, therefore, the eqn. (2.3) takes the form

$$R^h_{ijk,a} - \lambda_a R^h_{ijk} + \lambda_a C^h_{ijk} + \frac{(R_{,a} - \lambda_a R)}{n(n+2)} \{g_{ik} \delta^h_j - g_{jk} \delta^h_i + F_{ik} F^h_j - F_{jk} F^h_i + 2F_{ij} F^h_k\} = 0. \qquad ...(3.2)$$

If the space is Kaehlerian Ricci-recurrent then the above equation reduces to

$$R^h_{ijk,a} + \lambda_a (C^h_{ijk} - R^h_{ijk}) = 0. \qquad ...(3.3)$$

Conversely, if the eqn. (3.3) holds, then we have

$$\frac{(R_{,a} - \lambda_a R)}{n(n+2)} (g_{ik} \delta^h_j - g_{jk} \delta^h_i + F_{ik} F^h_j - F_{jk} F^h_i + 2F_{ij} F^h_k) = 0. \qquad ...(3.4)$$

Since $R = R_{ij} g^{ij}$, we have immediately from the above equation that

$$R_{ij,a} - \lambda_a R_{ij} = 0$$

which shows that the space is Kaehlerian Ricci-recurrent.

Theorem 3.3 — In a Kaehlerian concircular symmetric space the scalar curvature is constant.

PROOF : From eqns. (1.9) and (3.1), we obtain

$$R^h_{ijk,a} + \frac{R_{,a}}{n(n+2)} (g_{ik} \delta^h_j - g_{jk} \delta^h_i + F_{ik} F^h_j - F_{jk} F^h_i + 2F_{ij} F^h_k) = 0. \qquad ...(3.5)$$

Contracting indices h and k, we have

$$R_{ij,a} + \frac{R_{,a}}{n(n+2)} (F_{ih} F^h_j - F_{jh} F^h_i + 2F_{ij} F^h_h) = 0. \qquad ...(3.6)$$

Multiplying the above equation by g^{ij}, we get

$$R_{,a} + \frac{R_{,a}}{n(n+2)} \{g^{ij} (F_{ih} F^h_j - F_{jh} F^h_i + 2F_{ij} F^h_h)\} = 0 \qquad ...(3.7)$$

or

$$R_{,a} \left\{ 1 + \frac{g^{ij}}{n(n+2)} (F_{ih} F^h_j - F_{jh} F^h_i + 2F_{ij} F^h_h) \right\} = 0 \qquad ...(3.7a)$$

which implies

$$R_{,a} = 0 \qquad \ldots(3.8)$$

i.e. R is constant.

References

Lal, K. B., and Singh, S. S. (1971). On Kaehlerian space with recurrent Bochner curvature. *Acc. Naz. Dei Lincei*, Vol. LI, Nos. 3-4, 143-50.

Sinha, B. B. (1973). On *H*-curvature tensors in Kaehler manifold. *Kyungpook Math. J.*, **13**, No. 2, 185-89.

Tachibana, S. (1967). On the Bochner curvature tensor. *Nat. Sci. Report, Ochanomizu University*, **18**(1), 15-19.

Yano, K. (1965). Differential Geometry on Complex and Almost Complex Spaces. Pergamon Press, London, pp. 70-71.