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ONE-DIMENSIONAL PERTURBATIONS

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The purpose of this article is to study the change which results in the spectrum of a self-adjoint operator by the addition to it of a positive multiple of one-dimensional projection. The paper contains two theorems. A strict inequality between the eigenvalues of the operators involved is established in the first part of Theorem 1, whereas the second part, which follows immediately from the first part, is a special case of a classical result known as Sturm's separation Theorem (Rao 1965, p. 52). A far reaching generalization of the second part of Theorem 1 is contained in Kato (1966, p. 290). Theorem 2 contains a converse of Theorem 1.

We consider only operators on an n-dimensional Hilbert space $\mathcal{H}$. Let x be a normalized element of the Hilbert space and P the projection in the direction x. Relative to an orthonormal basis $\{e_i\}$, it is clear that P is represented by the matrix $P_{ij} = (e_j, x)(x, e_i)$.

The properties of the class of functions $V(\zeta)$ analytic in the upper half plane with positive imaginary part: $V(\zeta) = u(\zeta) + iv(\zeta)$, $v(\zeta) \geqslant 0$, may be found in Aronszajn and Donoghue (1957). In particular a function is in this class if and only if

$$V(\zeta) = \alpha\zeta + \beta + \int \left[\frac{1}{\lambda - \zeta} - \frac{\lambda}{\lambda^2 + 1}\right] d\mu$$

where $\alpha \geqslant 0$. β is real and μ is a positive Borel measure on the real axis for which $\int (\lambda^2 + 1)^{-1} d\mu(\lambda)$ is finite. The representation is unique.

Theorem 1 — Let A be a self-adjoint operator acting on the n-dimensional space with eigenvalues $\{\lambda_i\}$ and for some positive t, let B be given by $B = A + tP$. Assume that none of the numbers (x, e_j) $(j = 1, \ldots, n)$ vanishes, where $\{e_j\}$ is an orthonormal basis of eigenvectors of A. If the eigenvalues of B are the numbers $\{\mu_j\}$, then

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \ldots < \lambda_n < \mu_n.$$

In the general case, where x may be orthogonal to one or more e_j, we have

$$\lambda_1 \leqslant \mu_1 \leqslant \lambda_2 \leqslant \mu_2 \leqslant \ldots \leqslant \lambda_n \leqslant \mu_n.$$

PROOF : Obviously $B \geqslant A$ since

$$((B - A)\, y, y) = (tPy, y) = (t(y, x)\, x, y) = t \mid (x, y) \mid^2 \geqslant 0.$$

We introduce the resolvents of A and B :

$$R_\zeta = (A - \zeta I)^{-1},\ R'_\zeta = (B - \zeta I)^{-1}.$$

If u is an arbitrary element of $\mathcal{H}$, then

$$R'_\zeta u = v \text{ (say)}.$$

Now
$$\begin{aligned} u &= (B - \zeta I)\, v \\ &= (A - \zeta I)\, v + tPv \\ &= (A - \zeta I)\, v + t(v, x)\, x \end{aligned}$$

and applying the resolvent R_ζ, we obtain

$$R_\zeta u = v + t(v, x)\, R_\zeta x$$

and

$$v = R'_\zeta\, u = R_\zeta u - t(v, x)\, R_\zeta x.$$

Taking the inner product with x,

$$(v, x) = (R_\zeta u, x) - t(v, x)\,(R_\zeta x, x).$$

Solving for (v, x), we get

$$(v, x) = \frac{(R_\zeta u, x)}{1 + t(R_\zeta x, x)}.$$

Substituting for (v, x), we obtain

$$R'_\zeta u = R_\zeta u - \frac{t(R_\zeta u, x)}{1 + t(R_\zeta x, x)}\, R_\zeta x.$$

Moreover,

$$\begin{aligned} (R_\zeta x, x) &= \left((A - \zeta I)^{-1} \sum_{j=1}^{n} (x, e_j)\, e_j,\ \sum_{k=1}^{n} (x, e_k)\, e_k\right) \\ &= \sum_{j=1}^{n} \mid (x, e_j) \mid^2 ((A - \zeta I)^{-1} e_j, e_j) \\ &= \sum_{j=1}^{n} \frac{\mid (x, e_j) \mid^2}{\lambda_j - \zeta}. \end{aligned}$$

Hence,

$$R'_\zeta u = R_\zeta u - \frac{t(R_\zeta u, x)}{\phi(\zeta)} R_\zeta x$$

where

$$\phi(\zeta) = 1 + t(R_\zeta x, x) = 1 + t \sum_{j=1}^{n} \frac{|(x, e_j)|^2}{\lambda_j - \zeta}.$$

Now any zero of $\phi(\zeta)$ is a pole of R'_ζ, hence an eigenvalue of B.

Our hypothesis that none of the numbers (x, e_j) vanished makes sure that $\phi(\zeta)$ has fully n real zeros, one in evey interval of the form $(\lambda_k, \lambda_{k+1})$ as well as one on the right of λ_n. Accordingly, under the hypothesis (x, e_j) vanishes for no j ($j = 1, ..., n$), we have

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < ... < \lambda_n < \mu_n.$$

In the general case when x is orthogonal to one or more e_j ($j = 1, 2, ..., n$), we perturb P to obtain P' so that if P' is the projection in the direction of x', where x' is a normalized element of the Hilbert space, then (x', e_j) vanishes for no j ($j = 1, 2 ..., n$). If $\{\mu'_j\}$ are the eigenvalues of $B' = A + tP'$, then

$$\lambda_1 < \mu'_1 < \lambda_2 < \mu'_2 < ... < \lambda_n < \mu'_n.$$

In the limit as $P' \to P$, we have

$$\lambda_1 \leqslant \mu_1 \leqslant \lambda_2 \leqslant \mu_2 \leqslant ... \leqslant \lambda_n \leqslant \mu_n.$$

Since P is a projection on a one-dimensional subspace, its trace is $+1$. Now $Tr(B) = \Sigma\, \mu_k = Tr(A + tP) = Tr(A) + t = \Sigma\, \lambda_k + t$, hence $t = \Sigma\, (\mu_k - \lambda_k)$. The rational function $\phi(\zeta)$ is regular at infinity and equal to $+1$ there, so it may also be written as a product

$$\phi(\zeta) = \prod_{k=1}^{n} \frac{\zeta - \mu_k}{\zeta - \lambda_k} \qquad ...(1)$$

It is worthwhile to consider a converse to the previous theorem.

Theorem 2 — Let $\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < ... < \lambda_n < \mu_n$ and A a self-adjoint operator on the n-dimensional space having the λ_i for eigenvalues. Then there exists a normalized element x and a corresponding one-dimensional projection P such that for an appropriate $t > 0$ the operator $B = A + tP$ has the eigenvalues μ_i.

PROOF: We consinder the function $\phi(\zeta)$ given by (1) and show that it is in the Pick class. Between any two successive poles λ_k, λ_{k+1} there is precisely one zero μ_k

and that zero is simple. Now, if the residue of $\phi(\zeta)$ at those poles were of different sign, the function, near the ends of the interval $(\lambda_k, \lambda_{k+1})$ would be large in absolute value and would have the same sign near the poles. Thus there would be an even number of zeros in between, multiplicity counted. Since this is not the case, the residues have the same sign at every pole. Now it is fairly easy to determine that sign : the function $\phi(\zeta)$ is clearly positive and smaller than 1 for real $\zeta > \mu_n$, while it is equal to $+1$ at infinity. Thus the function increases, and all the residues are negative. It is, therefore, a Pick function. We may write

$$\phi(\zeta) = 1 + \sum \frac{m_k}{\lambda_k - \zeta}$$

where the masses m_k are positive. It is also easy to see that

$$\lim_{x \to \infty} x(1 - \phi(x)) = \Sigma\, m_k$$

and this limit can also be computed from the product representation of $\phi(\zeta)$; we find

$$\Sigma\, m_k = \Sigma\, \mu_k - \lambda_k.$$

Call this quantity t and write $\phi(\zeta) = 1 + t \sum \frac{m'_k}{\lambda_k - \zeta}$.

Evidently $\Sigma\, m'_k = 1$ and so if we choose a vector $x = \Sigma\, a_k e_k$ where e_k are the normalized eigenvectors of A and a_k any solution of $a_k^2 = m'_k$, then x is normalized. The operator $B = A + tP$ where P is projection associated with x is then associated with the function $\phi(\zeta)$ and has the prescribed μ_k for its eigenvalues. Our argument shows that there is a variety of one-dimensional projections P so that $A + tP$ has the required spectrum. Clearly a canonical choice is the simplest one, namely, the one in which every component of x relative to a fixed basis of eigenvectors of A is positive. In this case a_k is the positive square root of m'_k.

Further details concerning this canonical choice have been found by Loewner. Since these details cannot easily be found in the literature, we give them here.

We introduce the polynomials $P(\zeta) = \prod_{k=1}^{n} (\zeta - \mu_k)$ and $Q(\zeta) = \prod_{k=1}^{n} (\zeta - \lambda_k)$ and write $\phi(\zeta) = \frac{P(\zeta)}{Q(\zeta)}$. The negative of the residue of this function at λ_k is the positive number

$$m_k = -\lim \frac{(\zeta - \lambda_k)\, P(\zeta)}{Q(\zeta)} = -\lim \frac{P(\zeta)}{[Q(\zeta) - Q(\lambda_k)]/(\zeta - \lambda_k)} = -\frac{P(\lambda_k)}{Q'(\lambda_k)}.$$

Similarly, the function $\psi(\zeta) = -1/\varphi(\zeta)$ is also in the Pick class; it can be written as $-Q/P = -1 + \Sigma\, m_k^*/(\mu_k - \zeta)$, where $0 < m_k^* = Q(\mu_k)/P'(\mu_k)$. We will put $c_k = \sqrt{m_k}$, where we take the positive square root; similarly c_k^* is the positive square root of m_k^*. Next we introduce a matrix W with real elements

$$w_{ij} = \frac{c_i^* c_j}{(\mu_i - \lambda_j)},$$

and verify that W is unitary, as follows.

For $i \neq k$,

$$\sum w_{ij} w_{jk}^* = \sum_j w_{ij} w_{kj} = \sum_j \frac{c_i^* c_j c_k^* c_j}{(\mu_i - \lambda_j)(\mu_k - \lambda_j)}$$

$$= \frac{c_i^* c_k^*}{\mu_k - \mu_i} \sum_j m_j \left(\frac{1}{\mu_i - \lambda_j} - \frac{1}{\mu_k - \lambda_j} \right)$$

$$= \frac{c_i^* c_k^*}{\mu_k - \mu_i} [\{1 - \phi(\mu_i)\} - \{1 - \phi(\mu_k)\}] = 0.$$

For $i = k$,

$$\sum_j w_{ij}^2 = \sum_j \frac{c_i^{*2} c_j^2}{(\mu_i - \lambda_j)^2} = c_i^{*2} \sum_j \frac{m_j}{(\mu_i - \lambda_j)^2}$$

$$= m_i^* \phi'(\mu_i) = \frac{Q(\mu_i)}{P'(\mu_i)} \left[\frac{Q(\mu_i)\, P'(\mu_i) - P(\mu_i)\, Q'(\mu_i)}{Q^2(\mu_i)} \right] = 1.$$

Now let M be the diagonal marix with eigenvalues μ_i. We form $B = W^{-1}MW$ to obtain a symmetric matrix having the μ_i as its eigenvalues. Consider

$$(W^{-1}MW)_{ij} = \sum_k w_{ik}^* \mu_k w_{kj}$$

$$= \sum_k w_{ki} w_{kj} \mu_k$$

$$= \sum_k \frac{c_k^* c_i}{\mu_k - \lambda_i} \frac{c_k^* c_j}{\mu_k - \lambda_j} \mu_k$$

$$= c_i c_j \sum_k \frac{m_k^* \mu_k}{(\mu_k - \lambda_i)(\mu_k - \lambda_j)}.$$

So for $i \neq j$,

$$(B - A)_{ij} = c_i c_j \sum_k \frac{m_k^* \mu_k}{\lambda_i - \lambda_j} \left[\frac{1}{\mu_k - \lambda_i} - \frac{1}{\mu_k - \lambda_j} \right]$$

$$= \frac{c_i c_j}{\lambda_i - \lambda_j} \left[\sum m_k^* \frac{\mu_k - \lambda_i + \lambda_i}{\mu_k - \lambda_i} - \sum m_k^* \frac{\mu_k - \lambda_j + \lambda_j}{\mu_k - \lambda_j} \right]$$

$$= \frac{c_i c_j}{\lambda_i - \lambda_j} \left[\sum m_k^* - \sum m_k^* + \lambda_i \sum \frac{m_k^*}{\mu_k - \lambda_i} - \lambda_j \sum \frac{m_k^*}{\mu_k - \lambda_j} \right]$$

$$= \frac{c_i c_j}{\lambda_i - \lambda_j} [\lambda_i(\psi(\lambda_i) + 1) - \lambda_j(\psi(\lambda_j) + 1)]$$

$$= \frac{c_i c_j}{\lambda_i - \lambda_j} (\lambda_i - \lambda_j) = c_i c_j.$$

For $i = j$,

$$(B - A)_{ij} = c_i^2 \sum_k \frac{m_k^* \mu_k}{(\mu_k - \lambda_i)^2} - \lambda_i.$$

Now

$$c_i^2 \sum_k \frac{m_k^* \mu_k}{(\mu_k - \lambda_i)^2} = c_i^2 \sum_k m_k^* \left[\frac{1}{(\mu_k - \lambda_i)} + \frac{\lambda_i}{(\mu_k - \lambda_i)^2} \right]$$

$$= c_i^2 \sum_k \frac{m_k^*}{\mu_k - \lambda_i} + \lambda_i c_i^2 \sum_k \frac{m_k^*}{(\mu_k - \lambda_i)^2}$$

$$= c_i^2 (1 + \psi(\lambda_i)) + \lambda_i c_i^2 \psi'(\lambda_i)$$

$$= c_i^2 + \lambda_i.$$

Since $\psi'(\lambda_i) = 1/c_i^2 \cdot$

Hence for $i = j$,

$$(B - A)_{ij} = c_i^2 \cdot$$

Because $(B - A)_{ij} = c_i c_j$, where the numbers c_i are positive it follows that $B = A + tP$ where $t > 0$ and P is the projection associated with the normalized vector x where $x_i = (x, e_i)$.

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