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SOME RESULTS ON PERIODIC AND FIXED POINTS*

ASHOK K. SHARMA

Department of Mathematics, G.D. Salwan College (University of Delhi), Rajinder Nagar, Delhi 110060

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Some results concerning the existence of common periodic points and common fixed points of mappings which are not necessarily continuous have been obtained. The conditions under which these points are unique have also been discussed.

§1. The well-known Banach contraction principle states that if (X, d) is a complete metric space and f, a contraction on X, then f has a unique fixed point in X and for each $x \in X$, the sequence of iterates $\{f^n(x)\}$ converges to this fixed point. This principle has been generalized in several ways by various authors (see for example Fukushima 1971, Gupta 1972, Hardy and Rogers 1973, Jaggi 1977, Sehgal 1969 and Zamfirescu 1972). We have proved (Sharma 1976) the following result which generalizes Theorem 1 of Sehgal (1969).

Theorem A — Let f be a continuous mapping on a complete metric space (X, d) such that for each $x \in X$, there exists a positive integer $m(x)$ satisfying

$$d(f^{m(x)}(x), f^{m(x)}(y)) \leqslant a_1 d(x, f^{m(x)}(x)) + a_2 d(y, f^{m(x)}(y)) + a_3 d(x, f^{m(x)}(y)) + a_4 d(y, f^{m(x)}(x)) + a_5 d(x, y),$$

for all $x, y \in X$ and for positive real constants a_i $(i = 1, 2, \ldots, 5)$ with

$$a_1 + a_2 + 2a_3 + 2a_4 + a_5 < 1.$$

Then f has a unique fixed point $u \in X$ and $f^n(x) \to u$ for each $x \in X$.

In this paper too, using the computations based on the techniques adopted in Theorem A and Theorems 1 and 2 of Fukushima (1971) some results on the existence of common fixed points and common periodic points have been proved. These results turn out to be the generalizations of the results obtained by Fukushima (1971), Gupta (1972) and many others. It may be remarked that the fixed points obtained by Fukushima (1971) and Gupta (1972) are not necessarily unique, whereas in the present paper, we have also discussed the conditions under which the fixed points are unique.

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Let f_1, f_2 be two mappings on a metric space (X, d). Before proving our results we recall the following :

Definition (Sharma 1979) — By an orbit $x_0 \in X$ with respect to f_1 and f_2 we mean a sequence $O(x_0, f_1, f_2) = \{x_n : x_n = f_1 x_{n-1}$ or $f_2 x_{n-1}$, according as n is odd or even$\}$. Further, the space X is said to be (x_0, f_1, f_2)—jointly orbitally complete if every Cauchy sequence in the orbit $O(x_0, f_1, f_2)$ is convergent in X.

Also, for the case $f_1 = f_2 = f$, X is said to be (x_0, f)—orbitally complete (refer Jaggi 1977).

§2. We begin with the following :

Theorem 1 — Let f_1, f_2 be two mappings on a complete metric space (X, d) and $a_i(x, y)$ $(i = 1, 2, \ldots, 6)$ be real-valued symmetric and bounded functions on $X \times X$ with $a_1 \geqslant 0$, $a_5 \geqslant 0$ and $a_6 \geqslant 0$. If for each $x, y \in X$, there exist positive integers $m(x)$ and $m(y)$ such that for $p, q \in \{0, 1\}$,

$$a_1(x, y)\, d(f_1^{m(x)+p}(x), f_2^{m(y)+q}(y)) + a_2(x, y)\, d(x, y)$$
$$\leqslant a_3(x, y)\, d(x, f_1^{m(x)+p}(x)) + a_4(x, y)\, d(y, f_2^{m(y)+q}(y)))$$
$$+ a_5(x, y)\, d(x, f_2^{m(y)+q}(y) + a_6(x, y)\, d(y, f_1^{m(x)+p}(x) \quad \ldots(1)$$

where

$$0 < s \leqslant a_1(x, y) - a_4(x, y) - a_5(x, y) \quad \ldots(2)$$

$$0 < s \leqslant a_1(x, y) - a_3(x, y) - a_6(x, y) \quad \ldots(3)$$

$$a_2(x, y) < a_3(x, y) + a_5(x, y) \leqslant a_4(x, y) + a_5(x, y)$$
$$\leqslant \frac{K_1 a_1(x, y) + a_2(x, y)}{K_1 + 1};\ K_1 \in (0, 1) \quad \ldots(4)$$

$$a_2(x, y) < a_3(x, y) + a_6(x, y) \leqslant a_4(x, y) + a_6(x, y)$$
$$\leqslant \frac{K_2 a_1(x, y) + a_2(x, y)}{K_2 + 1};\ K_2 \in (0, 1) \quad \ldots(5)$$

then there exists at least one common fixed point of f_1 and f_2.

PROOF : Let $x_0 \in X$. Define a sequence $\{x_n\}$ in X by

$$x_n = \begin{cases} f_1^{m(x_{n-1})}(x_{n-1}), & n \text{ is odd} \\ f_2^{m(x_{n-1})}(x_{n-1}), & n \text{ is even.} \end{cases}$$

Now, using (1), (2) and (4) for $x = x_{2n}$, $y = x_{2n+1}$, $p = q = 0$ and denoting $a_i^{2n} = a_i(x_{2n}, x_{2n+1})$ $(i = 1, 2, \ldots, 6)$, we get

$$d(x_{2n+1}, x_{2n+2}) \leqslant \frac{a_3^{2n} + a_5^{2n} - a_2^{2n}}{a_1^{2n} - a_4^{2n} - a_5^{2n}} d(x_{2n}, x_{2n+1})$$

$$\leqslant \frac{[(K_1 a_1^{2n} + a_2^{2n})/(K_1 + 1)] - a_2^{2n}}{a_1^{2n} - [(K_1 a_1^{2n} + a_2^{2n})/(K_1 + 1)]} d(x_{2n}, x_{2n+1}).$$

Thus,

$$d(x_{2n+1}, x_{2n+2}) \leqslant K_1 d(x_{2n}, x_{2n+1}).$$

Similarly, by using (1), (3) and (5) for $x = x_{2n}$, $y = x_{2n-1}$ and $p = q = 0$, we get

$$d(x_{2n}, x_{2n+1}) \leqslant K_2 d(x_{2n-1}, x_{2n}).$$

Therefore,

$$d(x_{2n}, x_{2n+1}) \leqslant K_1^n K_2^n d(x_0, x_1)$$

and

$$d(x_{2n+1}, x_{2n+2}) \leqslant K_1^{n+1} K_2^n d(x_0, x_1).$$

Now, for $m = 2l$,

$$\begin{aligned} d(x_m, x_{m+n}) &\leqslant d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \ldots + d(x_{m+n-1}, x_{m+n}) \\ &\leqslant [K_1^l K_2^l + K_1^{l+1} K_2^l + \ldots + n \text{ terms}]\, d(x_0, x_1) \\ &< [K_1^l K_2^l + K_1^{l+1} K_2^l + \ldots]\, d(x_0, x_1) \\ &= [K_1^l K_2^l (1 + K_1 K_2 + \ldots) \\ &\qquad + K_1^{l+1} K_2^l (1 + K_1 K_2 + \ldots)]\, d(x_0, x_1) \\ &= K_1^l K_2^l (1 + K_1) \frac{1}{1 - K_1 K_2} d(x_0, x_1) \\ &\to 0, \text{ as } l \to \infty. \end{aligned}$$

Similarly, for $m = 2l + 1$ we have $d(x_m, x_{m+n}) \to 0$, as $l \to \infty$. Thus, $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d) and hence converges to some $z \in X$. Now put $x = x_{2n}$, $y = z$, $p = q = 0$ in (1) and denoting $a_i^n = a_i(x_n, z)$ $(i = 1, 2, \ldots, 6)$, we obtain

$$a_1^{2n}\, d(x_{2n+1}, f_2^{m(z)}(z)) + a_2^{2n}\, d(x_{2n}, z)$$
$$\leqslant a_3^{2n}\, d(x_{2n}, x_{2n+1}) + a_4^{2n}\, d(z, f_2^{m(z)}(z)) + a_5^{2n}\, d(x_{2n}, f_2^{m(z)}(z))$$
$$+ a_6^{2n}\, d(z, x_{2n+1}).$$

Adding $a_1^{2n}\, d(z, f_2^{m(z)}(z))$ to both the sides we get

$$(a_1^{2n} - a_4^{2n} - a_5^{2n})\, d(z, f_2^{m(z)}(z))$$
$$\leqslant a_1^{2n}\, [d(z, f_2^{m(z)}(z) - d(x_{2n+1}, f_2^{m(z)}(z)] + a_3^{2n}\, d(x_{2n}, x_{2n+1})$$
$$+ (a_5^{2n} - a_2^{2n})\, d(x_{2n}, z) + a_6^{2n}\, d(z, x_{2n+1})$$
$$\leqslant (a_1^{2n} + a_6^{2n})\, d(z, x_{2n+1}) + a_3^{2n}\, d(x_{2n}, x_{2n+1}) + (a_5^{2n} - a_2^{2n})\, d(x_{2n}, z).$$

As a_i's are bounded, by choosing n sufficiently large, the right-hand side can be made arbitrarily small and hence by (2)

$$d(z, f_2^{m(z)}(z)) = 0.$$

Similarly, on putting $x = z,\ y = x_{2n-1},\ p = q = 0$ and using (3) we obtain $d(z, f_1^{m(z)}(z)) = 0$. Thus

$$f_1^{m(z)}(z) = f_2^{m(z)}(z) = z.$$

We claim that z is a common fixed point of f_1 and f_2. For this, taking $x = y = z$, $p = 1$, $q = 0$ in (1) and denoting $a_i = a_i(z, z)$ $(i = 1, 2, \ldots, 6)$, we obtain

$$a_1 d(f_1^{m(z)+1}(z), f_2^{m(z)}(z)) + a_2 d(z, z)$$
$$\leqslant a_3 d(z, f_1^{m(z)+1}(z)) + a_4 d(z, f_2^{m(z)}(z)) + a_5 d(z, f_2^{m(z)}(z))$$
$$+ a_6 d(z, f_1^{m(z)+1}(z)).$$

Therefore, $(a_1 - a_3 - a_6)\, d(f_1(z), z)) \leqslant 0,$

and using (3) again we get $d(f_1(z), z) = 0$. Similarly, z is a fixed point of f_2. Hence z is a common fixed point of f_1 and f_2.

Remark 1 : The fixed point z obtained in Theorem 1 need not be a unique common fixed point of f_1 and f_2. But if in the hypothesis we also have

$$a_1(x, y) + a_2(x, y) > a_5(x, y) + a_6(x, y), \text{ for all } x, y \in X \qquad \ldots(6)$$

then z would be a unique fixed point of each of f_1 and f_2. It may also be remarked that with the present set of conditions we may omit $a_1 \geqslant 0$ for it results out from the other assumptions in the following way.

From (4)

$$a_2 < \frac{K_1 a_1 + a_2}{K_1 + 1} \Rightarrow a_2 < a_1$$

and then from (6), $a_5 \geqslant 0$ and $a_6 \geqslant 0$ we have

$$2a_1 > a_1 + a_2 > a_5 + a_6 \geqslant 0 \Rightarrow a_1 > 0.$$

Remark 2 : If the condition (1) holds for $p = 0$ and $q \in \{0, 1\}$ (or $q = 0$ and $p \in \{0, 1\}$), then f_1 and f_2 have a common periodic point z with the same period $m(z)$ and z would be a fixed point of atleast one of them. Moreover, if the condition (1) holds for $p = q = 0$ only, then the common periodic point z may fail to be a fixed point of either of them.

Remark 3 : For the case $m(x) = 1$, for each $x \in X$, the completeness of X can be replaced by (x_0, f_1, f_2)-jointly orbitally completeness of X for some $x_0 \in X$.

Remark 4 : By assuming $f_1 = f_2 = f$, we see that f has a fixed point z and that it will be unique if the condition (6) is also satisfied. Further, for the case $m(x) = 1$, for each $x \in X$, the completeness of X can be replaced by (x_0, f)-orbitally completeness of X for some $x_0 \in X$.

Now we state the following theorems which are generalizations of Theorem 1 of Fukushima (1971) and Theorem 4.7 of Gupta (1972). Their proofs run on the lines similar to that of Theorem 1. The conditions required in these cases are much weaker then those assumed in Remark 4.

Theorem 2 — Let f be a mapping on a complete metric space (X, d) such that for $q \in \{0, 1\}$

$$\begin{aligned} a_1(x, y)\, d(f^{m(x)}(x), f^{m(y)+q}(y)) + a_2(x, y)\, d(x, y) \\ \leqslant a_3(x, y)\, d(x, f^{m(x)}(x)) + a_4(x, y)\, d(y, f^{m(y)+q}(y)) \\ + a_5(x, y)\, d(x, f^{m(y)+q}(y)) + a_6(x, y)\, d(y, f^{m(x)}(x)) \quad \ldots(7) \end{aligned}$$

for all $x, y \in X$, where $m(x)$ and $a_i (i = 1, 2, \ldots, 6)$ are as defined in Theorem 1 with $a_1 \geqslant 0$. Further, if the conditions (2), (4) and $a_5 \geqslant 0$ (or the conditions (3), (5) and $a_6 \geqslant 0$) hold, then f has a fixed point in X. If in addition to these conditions we also have (6), then the fixed point of f would be unique and the condition $a_1 \geqslant 0$ may be replaced by $a_6 \geqslant 0$ (or $a_5 \geqslant 0$).

Theorem 3 — Let $\{f_n\}$ be a sequence of mappings on a complete metric space such that for $i \neq j$ and $q \in \{0, 1\}$

$$\begin{aligned} a_1(x, y)\, d(f_i^{m(x)}(x), f_j^{m(y)+q}(y)) + a_3(x, y)\, d(x, y) \\ \leqslant a_3(x, y)\, d(x, f_i^{m(x)}(x)) + a_4(x, y)\, d(y, f_j^{m(y)+q}(y)) \\ + a_5(x, y)\, d(x, f_j^{m(y)+q}(y)) + a_6(x, y)\, d(y, f_i^{m(x)}(x)) \quad \ldots(8) \end{aligned}$$

for all $x, y \in X$, where $m(x)$ and $a_i (i = 1, 2, ..., 6)$ are as defined in Theorem 1 with $a_1 \geqslant 0$. Further, if the conditions (2), (4) and $a_5 \geqslant 0$ (or the conditions (3), (5) and $a_6 \geqslant 0$) hold, then the sequence $\{f_n\}$ have a common fixed point z. If in addition to these conditions we also have (6), then z would be the unique fixed point of each of them and the condition $a_1 \geqslant 0$ may be replaced by $a_6 \geqslant 0$ (or $a_5 \geqslant 0$).

Remark 5 : If the condition (7) is satisfied for $q = 0$ only, then f has atleast one periodic point z with period $m(z)$ which is not necessarily a fixed point of f.

Corollary 1 — On assuming $a_5(x, y) = a_6(x, y) = 0$, Theorem 2 reduces to Theorem 4.7 of Gupta (1972).

Corollary 2 — If we take $a_5(x, y) = a_6(x, y) = 0$ and $a_3(x, y) = a_4(x, y)$, in Theorem 2, then Theorem 1 of Fukushima (1971) is deduced.

Remark 6 : It may be observed from Corollary 1 and Corollary 2 that not only Theorem 1, Theorem 2 is also a generalization of the results of Fukushima (1971) and Gupta (1972). Moreover, it may be noted that by assuming the functions $a_i(x, y)$ $(i = 1, 2, ..., 6)$ as constants and taking their particular values in Theorem 1 and Theorem 2, some important results of Hardy and Rogers (1973) Kannan (1968), Reich (1971), Zamfirescu (1972) and many others are obtained.

The following result guarantees the existence of a fixed point without assuming even the completeness of the metric space and thus generalizes Theorem 2 of Fukushima (1971) and Theorem 4.8 of Gupta (1972).

Theorem 4 — Let f be a mapping on a metric space (X, d) and let for each $x \in X$, there exist a positive integer $m(x)$ and a positive number $\epsilon(x)$ such that for each $y \in \{y : d(x, y) < \epsilon(x)\}$

$$\begin{aligned} a_1(x, y)\, d(f^{m(x)}(x), f^p(y)) + a_2(x, y)\, d(x, y) \\ \leqslant a_3(x, y)\, d(x, f^{m(x)}(x)) + a_4(x, y)\, d(y, f^p(y)) \\ + a_5(x, y)\, d(x, f^p(y)) + a_6(x, y)\, d(y, f^{m(x)}(x)) \end{aligned} \quad ...(9)$$

whenever $p \geqslant m(y)$, where $m(y)$ is the positive integer corresponding to y and $a_i (i = 1, 2, ..., 6)$, as defined in Theorem 1 satisfy the conditions (2), (3), $a_1 \geqslant 0$ and $a_5 \geqslant 0$ (or $a_6 \geqslant 0$). Further, if for some $x \in X$, the sequence $\{f^n(x)\}$ has a subsequence $\{f^{n_i}(x)\}$ converging to z then z is a fixed point of f and $\lim_{n \to \infty} f^n(x) = z$.

PROOF : By hypothesis there exists a positive integer $N(z, \delta(z))$ such that

$$d(z, f^{n_i}(x) < \delta(z), \text{ whenever } i > N(z, \delta(z)).$$

Let $p_i = \min \{p : m(f^{n_i}(x)) \leqslant p, f^{n_i+p}(x) = f^{n_j}(x) \in \{f^{n_i}(x)\}\}$. On putting $x = z$ and $y = f^{n_i}(x)$ in (9) and denoting $a_k^i = a_k(z, f^{n_i}(x))$ we get

$$a_1^i\, d(f^{m(z)}(z), f^{n_i+p_i}(x)) + a_2^i\, d(z, f^{n_i}(x))$$
$$\leqslant a_3^i\, d(z, f^{m(z)}(z)) + a_4^i\, d(f^{n_i}(x), f^{n_i+p_i}(x))$$
$$+ a_5^i\, d(z, f^{n_i+p_i}(x)) + a_6^i\, d(f^{n_i}(x), f^{m(z)}(z)).$$

Adding $a_1^i\, d(z, f^{m(z)}(z))$ to both the sides we get

$$(a_1^i - a_3^i - a_6^i)\, d(z, f^{m(z)}(z))$$
$$\leqslant (a_1^i + a_5^i)\, d(z, f^{n_j}(x)) + a_4^i\, d(f^{n_i}(x), f^{n_j}(x))$$
$$+ (a_6^i - a_2^i)\, d(z, f^{n_i}(x))$$

and as proved in Theorem 1, we see that $f^{m(z)}(z) = z$. Thus z is a periodic point of f with period $m(z)$. Now assuming $x = y = z$ and $p = m(z) + 1$ in (9) and proceeding as in the case of Theorem 1 we observe again that z is a fixed point of f.

We claim that $f^n(x) \to z$ as $n \to \infty$. For this, putting $x = z$ and $y = f^{n_i}(x)$ in (9) we have

$$a_1^i\, d(z, f^{n_i+p}(x)) + a_2^i\, d(z, f^{n_i}(x))$$
$$\leqslant a_4^i\, d(f^{n_i}(x), f^{n_i+p}(x)) + a_5^i\, d(z, f^{n_i+p}(x))$$
$$+ a_6^i\, d(f^{n_i}(x), z) \qquad \ldots(10)$$

for $p \geqslant m(f^{n_i}(x))$. Let $K = \max\{\,|a_i| : i = 1, 2, \ldots, 6\}$. Now there arise folloing three cases.

Case I: $a_4^i(z, f^{n_i}(x)) > 0$. Then (10) can be rewritten as

$$a_1^i\, d(z, f^{n_i+p}(x)) + a_2^i\, d(z, f^{n_i}(x))$$
$$\leqslant a_4^i\, [d(f^{n_i}(x), z) + d(z, f^{n_i+p}(x))] + a_5^i\, d(z, f^{n_i+p}(x))$$
$$+ a_6^i\, d(f^{n_i}(x), z).$$

Therefore,

$$d(z, f^{n_i+p}(x)) \leqslant \frac{a_4^i + a_6^i - a_2^i}{a_1^i - a_4^i - a_5^i}\, d(z, f^{n_i}(x))$$
$$\leqslant (3K/s)\, d(z, f^{n_i}(x)), \text{ for } p \geqslant m(f^{n_i}(x)).$$

Case II : $a_4^i(z, f^{n_i}(x)) = 0$ — This implies that

$$a_1(z, f^{n_i}(x)) - a_5(z, f^{n_i}(x)) \geqslant s > 0.$$

Thus

$$d(z, f^{n_i+p}(x) \leqslant \frac{a_6^i - a_2^i}{a_1^i - a_5^i} d(z, f^{n_i}(x))$$

$$\leqslant \frac{2K}{s} d(z, f^{n_i}(x)), \text{ for } p \geqslant m(f^{n_i}(x)).$$

Case III : $a_4^i(z, f^{n_i}(x)) < 0$ — Now

$$d(z, f^{n_i+p}(x)) \leqslant d(z, f^{n_i}(x)) + d(f^{n_i}(x), f^{n_i+p}(x)).$$

Therefore,

$$d(z, f^{n_i+p}(x)) \leqslant - \frac{a_4^i + a_2^i - a_6^i}{a_1^i - a_4^i - a_5^i} d(z, f^{n_i}(x))$$

$$\leqslant \frac{3K}{s} d(z, f^{n_i}(x), \text{ for } p \geqslant m(f^{n_i}(x)).$$

Now, since $\lim_{i \to \infty} f^{n_i}(x) = z$, all the three cases yield that $f^n(x) \to z$ as $n \to \infty$ and thus proving the result.

Remark 7: It may be noted that for different point x we have different fixed point z and thus $\{f^n(x)\}$ converges to different fixed points. But, if the condition (6) is also satisfied then, z is the unique fixed point of f and in this situation $f^n(x) \to z$ for each $x \in X$.

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