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SOME PLANE SYMMETRIC MODELS OF PERFECT FLUID DISTRIBUTION IN GENERAL RELATIVITY

S. R. Roy and Shyam Narain

Department of Mathematics, Banaras Hindu University, Varanasi 221005

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Plane symmetric models representing perfect fluid distribution have been obtained for which the free gravitational fields are of Petrov type D.

1. Introduction

Plane symmetric perfect fluid distribution was discussed by Taub (1956) in which flow was taken to be isentropic. Plane symmetric cosmological models representing perfect fluid have been discussed by Singh and Singh (1968), Singh and Abdussattar (1973) and Roy and Singh (1976). We consider in this paper plane symmetric models representing perfect fluid distribution for which the free gravitational field is of Petrov type D.

We consider the metric in the form

$$ds^2 = A^2(dt^2 - dx^2) - B^2dy^2 - C^2dz^2 \qquad \text{...(1)}$$

where metric potentials are functions of x and t alone. The energy momentum tensor for the perfect fluid distribution is given by

$$T_{ij} = (\rho + p)\, v_i v_j - p g_{ij} \qquad \text{...(2)}$$

together with

$$g^{ij} v_i v_j = 1 \qquad \text{...(3)}$$

ρ being the density, p the pressure and v_i the flow vector. The field equations are

$$-\,8\pi\, T_{ij} = R_{ij} - \tfrac{1}{2} R g_{ij}. \qquad \text{...(4)}$$

Equations (2) and (4) for the metric (1) lead to

$$v_2 = v_3 = 0.$$

The field equations (4) lead to

$$-\,8\pi\,[(\rho + p)(v_1^2) + pA^2] = \Big[\frac{B_{44}}{B} + \frac{C_{44}}{C} - \frac{1}{AB}(A_1B_1 + A_4B_4) - \frac{1}{AC}(A_1C_1 + A_4C_4) + \frac{1}{BC}(B_4C_4 - B_1C_1)\Big] \qquad \text{...(5)}$$

$$-8\pi[pB^2]=\frac{B}{A^2}\left[\frac{B}{A}(A_{44}-A_{11})+\frac{B}{C}(C_{44}-C_{11})+\frac{B}{A^2}(A_1^2-A_4^2)\right] \quad ...(6)$$

$$-8\pi[pC^2]=\frac{C}{A^2}\left[\frac{C}{A}(A_{44}-A_{11})+\frac{C}{B}(B_{44}-B_{11})+\frac{C}{A^2}(A_1^2-A_4^2)\right] \quad ...(7)$$

$$-8\pi[(\rho+p)(v_4^2)-pA^2]=\left[\frac{B_{11}}{B}+\frac{C_{11}}{C}+\frac{1}{BC}(B_1C_1-B_4C_4)-\frac{1}{AB}(A_1B_1+A_4B_4)-\frac{1}{AC}(A_1C_1+A_4C_4)\right] \quad ...(8)$$

$$-8\pi[(\rho+p)\,v_1v_4]=\left[\frac{B_{14}}{B}+\frac{C_{14}}{C}-\frac{1}{AB}(A_4B_1+A_1B_4)-\frac{1}{AC}(A_4C_1+A_1C_4)\right]. \quad ...(9)$$

From eqn. (3), we get

$$(v_4)^2-(v_1)^2=A^2. \quad ...(10)$$

In the above equations suffixes 1 and 4 after A, B and C denote partial differentiation with respect to x and t respectively.

Equations (5) to (10) are six equations in seven unknowns $A, B, C, \rho, p, v_1, v_4$. We, therefore, require an additional condition for complete determination of this set. We assume that the metric is of Petrov type II non-degenerate. This requires that

$$C_{(1212)}-C_{(1313)}=2\,C_{(1224)}. \quad ...(11)$$

In particular if $C_{(1224)}$ is zero the metric will be of Petrov type D. Equation (11) leads to

$$\frac{B_{11}+B_{44}+2B_{14}}{B}-\frac{C_{11}+C_{44}+2C_{14}}{C}=\frac{2}{AB}(A_1+A_4)(B_1+B_4)-\frac{2}{AC}(A_1+A_4)(C_1+C_4). \quad ...(12)$$

From eqns. (12), (6) and (7), and eqns. (5), (6), (8), (9) and (10), we get

$$\frac{B_{uu}}{B}-\frac{C_{uu}}{C}=2\frac{A_u}{A}\left(\frac{B_u}{B}-\frac{C_u}{C}\right) \quad ...(13)$$

$$\frac{B_{uv}}{B}=\frac{C_{uv}}{C} \quad ...(14)$$

and

$$\left[\frac{A_{uv}}{A}+\frac{A_uA_v}{A^2}+\frac{B_{uu}+2B_{uv}+B_{vv}}{4B}+\frac{C_{uu}-2C_{uv}+C_{vv}}{4C}\right.$$
$$\left.-\frac{A_uB_u+A_vB_v}{2AB}-\frac{A_uC_u+A_vC_v}{2AC}-\frac{B_uC_v+B_vC_u}{2BC}\right]$$
$$\times\left[\frac{B_{uu}+2B_{uv}+B_{vv}}{4B}+\frac{C_{uu}-2C_{uv}+C_{vv}}{4C}+\frac{B_uC_v+B_vC_u}{2BC}\right.$$
$$\left.-\frac{A_uB_u+A_vB_v}{2AB}-\frac{A_uC_u+A_vC_v}{2AC}-\frac{A_{uv}}{A}-\frac{A_uA_v}{A^2}\right]$$
$$=\left[\frac{B_{uu}-B_{vv}}{4B}+\frac{C_{uu}-C_{vv}}{4C}-\frac{A_uB_u-A_vB_v}{2AB}-\frac{A_uC_u-A_vC_v}{2AC}\right]^2 \quad ...(15)$$

where $u=\frac{1}{2}(x+t)$ and $v=\frac{1}{2}(x-t)$.

2. Solutions of the Field Equations

We assume that B and C are functions of A alone. Equations (13) and (14) respectively lead to

$$\frac{(B''/B)-(C''/C)}{(B'/B)-(C'/C)}=\frac{2A_u^2-AA_{uu}}{AA_u^2} \quad ...(16)$$

and

$$\frac{(B''/B)-(C''/C)}{(B'/B)-(C'/C)}=-\frac{A_{uv}}{A_uA_v} \quad ...(17)$$

where a prime indicates differentiation with respect to A.

From eqns. (16) and (17), we have

$$A^2\frac{A_v}{A_u}=H(v) \quad ...(18)$$

where $H(v)$ is an arbitrary function of v alone. From eqn. (17), we have

$$\frac{A_{uv}}{A_nA_v}=F(A) \quad ...(19)$$

where F is an unknown function of A. Equation (19) leads to

$$A=A\{\alpha(u)+\beta(v)\} \quad ...(20)$$

where α and β are functions of u and v respectively. From eqns. (18) and (20), we get

$$A^2=\frac{H(v)}{\beta_v}\alpha_u. \quad ...(21)$$

Differentiating (21) partially with respect to u and v respectively, we get

$$\frac{\alpha_{uu}}{\alpha_u^2} = \frac{(H/\beta_v)_{,v}}{(H/\beta_v)} \cdot \frac{1}{\beta_v} = -k$$

where k is a constant. Hence, we get

$$\alpha_u = e^{-k\alpha + L'} \qquad \text{...(22)}$$

and

$$\left(\frac{H}{\beta_v}\right) = e^{-k\beta + N'} \qquad \text{...(23)}$$

L' and N' being constants of integration. From eqn. (22), we get

$$\alpha = \frac{1}{L''} \log (mu + n) \qquad \text{...(24)}$$

where L'', m and n are arbitrary constants. From (21), (22) and (23), we get

$$A = \sigma e^{-\epsilon(\alpha+\beta)} \qquad \text{...(25)}$$

where $\sigma = e^{(L'+H')/2}$ and $\epsilon = k/2$. From eqn. (25), we get

$$\frac{A_{uv}}{A_u A_v} = \frac{1}{A}. \qquad \text{...(26)}$$

From eqns. (17) and (26), we get

$$B'C - BC' = \frac{\xi}{A} \qquad \text{...(27)}$$

ξ being an arbitrary constant. From eqns. (16), (17) and (26), we get

$$A_{uu} = \frac{3}{A} A_u^2. \qquad \text{...(28)}$$

We now assume that

$$A_{vv} = \frac{3}{A} A_v^2. \qquad \text{...(29)}$$

Putting $\nu = BC$ and $\mu = B/C$, we get from eqn. (27)

$$\frac{\mu'}{\mu} = \frac{\xi}{A\nu}. \qquad \text{...(30)}$$

From eqns. (15), (28), (29) and (30), we get

$$\{A(A\nu')'\}^2 - A(A\nu') \frac{1}{\nu} (A^2\nu'^2 - \xi^2) + 4(A^2\nu'^2 - \xi^2) - 16\nu^2 = 0. \qquad \text{...(31)}$$

Putting $A\nu' = F(\nu)$ in eqn. (31), we get

$$\frac{dF^2}{d\nu} = \frac{(F^2 - \xi^2)}{\nu} \mp \left[\frac{(F^2 - \xi^2)}{\nu} - 8\nu\right]. \qquad \ldots(32)$$

Case I — Taking the upper sign in (32), we get

$$F = \sqrt{4\nu^2 + L} \qquad \ldots(33)$$

L being the constant of integration.

Subcase I(a) — $L > 0$: Hence

$$\nu = \tfrac{1}{2}\, QA^2 - \frac{L}{8\, QA^2} \qquad \ldots(34)$$

Q being a positive constant. From eqns. (30) and (34), we get

$$\mu = k' \left[\frac{QA^2 - a}{QA^2 + a}\right]^{\xi/2a} \qquad \ldots(35)$$

where k' is an arbitrary constant and $a = \frac{\sqrt{L}}{2}$. From eqns. (34) and (35), we get

$$B = \frac{k_1}{A} \cdot \frac{(A^2 - \theta)^\lambda}{(A^2 + \theta)^{\lambda-1}} \qquad \ldots(36)$$

and

$$C = \frac{k_2}{A} \cdot \frac{(A^2 + \theta)^{\lambda-1}}{(A^2 - \theta)^\lambda} \qquad \ldots(37)$$

where $\lambda = \frac{\xi + 2a}{4a}$, $k_1 = \sqrt{\frac{k'Q}{2}}$, $k_2 = \frac{Q}{2k_1}$, $\theta = \frac{a}{Q}$. The metric (1) reduces to the form

$$ds^2 = A^2(dt^2 - dx^2) - \left[\frac{k_1}{A} \cdot \frac{(A^2 - \theta)^\lambda}{(A^2 + \theta)^{\lambda-1}}\right]^2 . dy^2$$
$$- \left[\frac{k_2}{A} \cdot \frac{(A^2 + \theta)^{\lambda-1}}{(A^2 - \theta)^\lambda}\right]^2 . dz^2. \qquad \ldots(38)$$

where A is given by eqn. (25).

Subcase I(b) — $L = 0$. Hence, we get

$$B = \theta_1 A e^{-\lambda_1/A^2} \qquad \ldots(39)$$

and

$$C = \frac{A}{2\theta_1}\, e^{\lambda_1/A^2} \qquad \ldots(40)$$

where θ_1 is an arbitrary constant and $\lambda_1 = \frac{\xi}{4Q}$. The metric (1) reduces to the form

$$ds^2 = A^2(dt^2 - dx^2) - \theta_1^2 A^2 e^{-2\lambda_1/A^2} \, . \, dy^2 - \frac{A^2}{4\theta_1^2} e^{2\lambda_1/A^2} \, . \, dz^2. \quad \ldots(41)$$

Subcase I(c) — $L < 0$. Hence, we get

$$\nu = \frac{a_1 A^4 + a_2}{A^2} \quad \ldots(42)$$

where a_1 is a positive constant and $a_2 = \frac{L}{16 a_1}$. From eqns. (30) and (42), we get

$$\mu = I_0 \exp\left(I_1 \tan^{-1}\left(\frac{A^2}{a_0}\right)\right) \quad \ldots(43)$$

where I_0 is an arbitrary constant and $I_1 = \frac{\xi}{2a_2}$, $a_0 = \frac{a_2}{a_1}$. From eqns. (42) and (43), we get

$$B^2 = a_1 I_0 \frac{(A^4 + a_0^2)}{A^2} \exp\left(I_1 \tan^{-1} \frac{A^2}{a_0}\right) \quad \ldots(44)$$

and

$$C^2 = \frac{a_1}{I_0} \cdot \frac{(A^4 + a_0^2)}{A^2} \exp\left(-I_1 \tan^{-1} \frac{A^2}{a_0}\right). \quad \ldots(45)$$

The metric (1) reduces to the form

$$ds^2 = A^2(dt^2 - dx^2) - \frac{a_1 I_0 (A^2 + a_0^2)}{A^2} \exp\left(I_1 \tan^{-1} \frac{A^2}{a_0}\right) . \, dy^2$$
$$- \frac{a_1(A^4 + a_0^2)}{I_1 A^2} \exp\left(-I_1 \tan^{-1} \frac{A^2}{a_0}\right) . \, dz^2. \quad \ldots(46)$$

Case II — Taking the lower sign in (32), we have

$$\log C_2 A = \int \frac{d\nu}{\sqrt{\nu^2 \log(C_1/\nu^8) + \xi^2}} \quad \ldots(47)$$

where C_1 is an arbitrary constant and C_2 the constant of integration. From eqn. (30), we get

$$\mu = \exp\left(\int \frac{\xi}{A\nu} dA + C_3\right) \quad \ldots(48)$$

C_3 being constant of integration. Thus, the metric (1) reduces to the form

$$ds^2 = A^2(dt^2 - dx^2) - \nu \exp\left(\int \frac{\xi}{A\nu} dA + C_3\right) . \, dy^2$$
$$- \nu \exp\left(-\int \frac{\xi}{A\nu} dA - C_3\right) . \, dz^2. \quad \ldots(49)$$

3. Some Physical Features

The expressions for pressure p, density ρ and v_1, v_4 the non-vanishing components of flow vector in different cases are as follows:

Case I(a)

$$p = \frac{h\beta_v}{mu + n}\left[\frac{A^4(L_1 - A^4) - \theta^4}{A^3(A^4 - \theta^2)^2}\right] \quad \ldots(50)$$

$$\rho = \frac{h\beta_v}{mu + n}\left[\frac{A^4(L_2A^4 + M_1) - 3\theta^4}{A^3(A^4 - \theta^2)^2}\right] \quad \ldots(51)$$

$$v_1 = \frac{A}{\sqrt{2}}\left[\frac{\{A^4(L_1 - A^4) - \theta^4\}\{l_1^2 + (mu + n)^2 \epsilon^2 \beta_v^2\}}{h\beta_v(mu + n)\{A^4(k_3A^4 + k_4) - 4\theta^4\}} - 1\right]^{1/2} \quad \ldots(52)$$

and

$$v_4 = \frac{A}{\sqrt{2}}\left[\frac{\{A^4(L_1 - A^4) - \theta^4\}\{l_1^2 + (mu + n)^2 \epsilon^2 \beta_v^2\}}{h\beta_v(mu + n)\{A^4(k_3A^4 + k_4) - 4\theta^4\}} + 1\right]^{1/2} \quad \ldots(53)$$

where

$$h = \frac{m\epsilon^2}{16\pi L''}, \qquad L_1 = 2\theta^2\{1 - 8\lambda(\lambda - 1)\}$$

$$L_2 = 13 - 16\lambda^2, \qquad M_1 = 2\theta^2(11 - 8\lambda)$$

$$k_3 = 4(3 - 4\lambda^2), \qquad k_4 = L_1 + M_1$$

$$l_1^2 = \frac{m^2\epsilon^2}{L''^2}.$$

Case I(b)

$$p = Q_1 \cdot \frac{\left\{5 + \frac{4\lambda_1^2}{A^4}\right\}}{A^2(mu + n)} \cdot \beta_v \quad \ldots(54)$$

$$\rho = Q_1 \cdot \frac{\left\{1 + \frac{4\lambda_1^2}{A^4}\right\}}{A^2(mu + n)} \cdot \beta_v \quad \ldots(55)$$

$$v_1 = \frac{A}{\sqrt{2}}\left[\left(\frac{A^4 + 4\lambda_1^2}{6A^4 + 8\lambda_1^2}\right)\left(\frac{l_1^2 + (mu + n)^2 \epsilon^2\beta_v^2}{\gamma_1(mu + n)}\right) - 1\right]^{1/2} \quad \ldots(56)$$

and

$$v_4 = \frac{A}{\sqrt{2}}\left[\left(\frac{A^4 + 4\lambda_1^2}{6A^4 + 8\lambda_1^2}\right)\left(\frac{l_1^2 + (mu + n)^2 \epsilon^2\beta_v^2}{\gamma_1(mu + n)}\right) + 1\right]^{1/2} \quad \ldots(57)$$

where

$$Q_1 = -\frac{m\epsilon^2\sigma^2}{8\pi L''}$$

and

$$\gamma_1 = \frac{m\epsilon^2\sigma^2}{L''}.$$

Case I(c)

$$p = \frac{h\beta_v}{mu+n}\left[\frac{8A^8 + r_1A^6 + r_2A^4 + r_3A^2}{A^3(A^4+a_0^2)}\right] \quad ...(58)$$

$$\rho = \frac{h\beta_v}{mu+n}\left[\frac{4A^8 + r_1A^6 + q_1A^4 + r_3A^2 - q_2}{A^3(A^4+a_0^2)}\right] \quad ...(59)$$

$$v_1 = \frac{A}{\sqrt{2}}\left[\left\{\frac{(A^4+a_0^2)(f_1A^2 - 2r_1A^6 - f_2A^4 - f_3)}{h\beta_v\, A(mu+n)}\right\} \times \left\{\frac{l_1^2 + (mu+n)^2\,\epsilon^2\,\beta_v^2}{12A^8 + 2r_1A^6 + (r_2+q_1)\,A^4 + 2r_3A^2 - q_2}\right\} - 1\right]^{1/2} \quad ...(60)$$

and

$$v_4 = \frac{A}{\sqrt{2}}\left[\left\{\frac{(A^4+a_0^2)(f_1A^2 - 2r_1A^6 - f_2A^4 - f_3)}{h\beta_v\, A(mu+n)}\right\} \times \left\{\frac{l_1^2 + (mu+n)^2\,\epsilon^2\beta_v^2}{12A^8 + 2r_1A^6 + (r_2+q_1)\,A^4 + 2r_3A^2 - q_2}\right\} + 1\right]^{1/2} \quad ...(61)$$

where

$$r_1 = 2I_1a_0, \qquad q_1 = (2 - I_1)\,4a_0^2$$

$$r_2 = 4I_1a_0^2 - 6I_1^2\,a_0^2 - 24a_0^2, \qquad q_2 = 6a_0^4$$

$$r_3 = 2I_1a_0^3, \qquad f_1 = 2a_0^2$$

$$f_2 = 12a_1a_0^2 + \frac{\xi^2}{2a_1}, \qquad f_3 = 4a_1a_0^4.$$

Case II

$$p = \frac{h\beta_v}{mu+n}\left[\frac{4 + (2 - \sqrt{\mu\nu})\log\dfrac{C_1}{\nu^8}}{\sqrt{\mu\nu}}\right] \quad ...(62)$$

$$\rho = \frac{h\beta_v}{mu+n}\left[\frac{12 - (3 - \frac{1}{2}\sqrt{\mu\nu})\log\dfrac{C_1}{\nu^8}}{\sqrt{\mu\nu}}\right] \quad ...(63)$$

$$v_1 = \frac{A}{\sqrt{2}} \left[\left\{ \frac{A\sqrt{\mu\nu}\left\{4\nu - \frac{1}{2}\nu \log \frac{C_1}{\nu^8} - \left(\frac{\nu+1}{\nu}\right)\sqrt{\nu^2 \log \frac{C_1}{\nu^8} + \xi^2}\right\}}{h\beta_v(mu+n)} \right\} \times \left\{ \frac{l_1^2 + (mu+n)^2\, \epsilon^2\beta_v^2}{16 - (1+\frac{1}{2}\sqrt{\mu\nu}) \log \frac{C_1}{\nu^8}} \right\} - 1 \right]^{1/2} \quad \ldots(64)$$

and

$$v_4 = \frac{A}{\sqrt{2}} \left[\left\{ \frac{A\sqrt{\mu\nu}\left\{4\nu - \frac{1}{2}\nu \log \frac{C_1}{\nu^8} - \left(\frac{\nu+1}{\nu}\right)\sqrt{\nu^2 \log \frac{C_1}{\nu^8} + \xi^2}\right\}}{h\beta_v(mu+n)} \right\} \times \left\{ \frac{l_1^2 + (mu+n^2)\, \epsilon^2\beta_v^2}{16 - (1+\frac{1}{2}\sqrt{\mu\nu}) \log \frac{C_1}{\nu^8}} \right\} + 1 \right]^{1/2} . \quad \ldots(65)$$

In all the above cases the space-time is of Petrov type *D*.

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