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ON KAEHLERIAN *S*-RECURRENT SPACES WITH BOCHNER CURVATURE TENSOR

S. K. D. DUBEY

Department of Mathematics, University of Gorakhpur, Gorakhpur 273001

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The purpose of the present paper is to define *S*-recurrent Kaehler space. Several theorems regarding Kaehlerian *S*-recurrent space, *S*-recurrent holomorphically projective Kaehler space, Kaehler space with a generalized parallel Bochner curvature tensor and Kaehlerian projective generalized symmetric space have been investigated.

1. INTRODUCTION

An $n(=2m)$ dimensional Kaehlerian space is a Riemannian space if it admits a structure tensor ϕ_i^h satisfying (Yano 1965)

$$\left.\begin{aligned} &\text{(a)} \quad \phi_i^h \phi_h^j = -\delta_i^j \\ &\text{(b)} \quad \phi_{ij} = -\phi_{ji} (\phi_{ij} = \phi_i^h \phi_{hj}) \\ &\text{(c)} \quad \nabla_j \phi_i^h = 0. \end{aligned}\right\} \quad \ldots(1.1)$$

The Riemannian curvature tensor R^h_{ijk} is given by

$$R^h_{ijk} = \partial_i \begin{Bmatrix} h \\ jk \end{Bmatrix} - \partial_j \begin{Bmatrix} h \\ ik \end{Bmatrix} + \begin{Bmatrix} h \\ il \end{Bmatrix} \begin{Bmatrix} l \\ jk \end{Bmatrix} - \begin{Bmatrix} h \\ jl \end{Bmatrix} \begin{Bmatrix} l \\ ik \end{Bmatrix}$$

whereas the Ricci tensor and the scalar curvature are respectively given by $R_{ij} = R^h_{hij}$ and $R = g^{ij}R_{ij}$. The Bochner curvature tensor K^h_{ijk} (with respect to real local coordinates) is defined (Tachibana 1967) as

$$\begin{aligned} K^h_{ijk} = R^h_{ijk} &+ \frac{1}{n+4}(R_{ik}\delta^h_j - R_{jk}\delta^h_i + g_{ik}R^h_j - g_{jk}R^h_i + S_{ik}\phi^h_j \\ &- S_{jk}\phi^h_i + \phi_{ik}S^h_j - \phi_{jk}S^h_i + 2S_{ij}\phi^h_k + 2\phi_{ij}S^h_k) \\ &- \frac{R}{(n+2)(n+4)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i + \phi_{ik}\phi^h_j - \phi_{jk}\phi^h_i + 2\phi_{ij}\phi^h_k) \end{aligned} \quad \ldots(1.2)$$

where

$$S_{jk} = \phi_j^h R_{hk}. \qquad \ldots(1.3)$$

The Bianchi identity can be written as (Yano 1965)

$$\nabla_l K_{kjih} + \nabla_k K_{jlih} + \nabla_j K_{lkih} = 0. \qquad \ldots(1.4)$$

The Kaehlerian spaces ($n > 2$) with parallel Bochner curvature tensor is given by (Matsumoto 1969)

$$\nabla_l K^h_{ijk} = 0. \qquad \ldots(1.5)$$

We shall call the Kaehlerian space satisfying (1.5), a PB_n-space and the space satisfying $K^h_{ijk} = 0$ will be called VB_n-space. Putting

$$\left.\begin{aligned} &\text{(a)} \quad \pi_{ij} = \frac{1}{n+4}\left(R_{ij} - \frac{R}{2(n+2)} g_{ij}\right) \\ &\text{(b)} \quad M_{ij} = \phi_i^h \pi_{hj} = \frac{1}{n+4}\left(S_{ij} - \frac{R}{2(n+2)} \phi_{ij}\right) \end{aligned}\right\} \qquad \ldots(1.6)$$

and

$$D^h_{ijk} = \pi_{ik}\delta^h_j - \pi_{jk}\delta^h_i - g_{ik}\pi^h_j - g_{jk}\pi^h_i + M_{ik}\phi^h_j - M_{jk}\phi^h_i$$
$$+ \phi_{ik}M^h_j - \phi_{jk}M^h_i + 2M_{ij}\phi^h_k + 2\phi_{ij}M^h_k \qquad \ldots(1.7)$$

eqn. (1.2) can be written as

$$K^h_{ijk} = R^h_{ijk} + D^h_{ijk} \cdot \qquad \ldots(1.8)$$

Also, we have

$$\left.\begin{aligned} &\text{(a)} \quad \pi = g^{ab}\pi_{ab} = \frac{R}{2(n+2)} \\ &\text{(b)} \quad M_{ij} = - M_{ji} \\ &\text{(c)} \quad D_{ijkh} = D_{khij} (D_{ijkh} = D^l_{ijk}\, g_{lh}). \end{aligned}\right\} \qquad \ldots(1.9)$$

A Kaehlerian space is said to be a recurrent space of first order if its curvature tensor R^h_{ijk} satisfies the relation

$$\nabla_l R^h_{ijk} = \Omega_l R^h_{ijk} \qquad \ldots(1.10)$$

for non-zero vector Ω_l. From (1.1c), (1.2), (1.4) and (1.10), we have

Theorem 1.1 — Every recurrent PB_n-space is VB_n-space.

Let a PB_n-space be an Einstein space, then the Ricci tensor satisfies

$$R_{ij} = \frac{R}{n} g_{ij}, \; \nabla_l R = 0 \qquad ...(1.11)$$

from which, we have

$$S_{ij} = \frac{R}{n} \phi_{ij}, \; \nabla_l R_{ij} = 0 \text{ and } \nabla_l S_{ij} = 0, \qquad ...(1.12)$$

which implies that $\nabla_l R^h_{ijk} = 0$ and gives:

Theorem 1.2 — Every Kaehlerian Einstein PB_n-space is symmetric in the sense of Cartan.

The holomorphically projective curvature tensor, given by

$$P^h_{ijk} = R^h_{ijk} + \frac{1}{n+2}(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}\phi^h_j - S_{jk}\phi^h_i + 2S_{ij}\phi^h_k) \qquad ...(1.13)$$

of a Kaehlerian space, which is invariant under any holomorphically projective correspondence (Tachibana 1967), corresponds to the Weyl's projective curvature tensor W^h_{ijk} of a Riemannian space, which is invariant under any projective correspondence.

2. Kaehlerian *S*-Recurrent Space

In analogy to Kaigorodov (1973), we give the following :

Definition 2.1 — A Kaehler space K_n satisfying the relation

$$(\nabla_{p_1} \nabla_{p_2} \cdots \nabla_{p_s}) K^h_{ijk} = \Omega_{p_1 p_2 \cdots p_s} K^h_{ijk} \qquad ...(2.1)$$

is said to be s-recurrent Kaehler space and

$\Omega_{p_1 p_2 \cdots p_s}$ ($\neq 0$) is s-recurrence tensor field.

Using eqns. (1.8) and (2.1), we have the following.

Theorem 2.1 — The Riemannian s-recurrent space implies Kaehlerian s-recurrent space if and only if the curvature tensor D^h_{ijk} is also s-recurrent with the same s-recurrence tensor field.

Theorem 2.1 can also be expressed as follows.

Theorem 2.2 — If the Riemannian curvature tensor is s-recurrent then the Bochner curvature tensor is also s-recurrent with the same recurrence tensor field but the converse is not true in general.

Theorem 2.3 — The recurrence tensor field and recurrence vector field Ω_{p_s} $(s \geqslant 2)$ of a Kaehler space satisfy the relation

$$\Omega_{p_1 p_2 \ldots p_s} = \{\nabla_{p_s} \Omega_{p_1 p_2 \ldots p_{s-1}} + \Omega_{p_1 p_2 \ldots p_{s-1}} \Omega_{p_s}\}. \qquad \ldots(2.2)$$

PROOF : For first-order Kaehler recurrent space, we have

$$\nabla_{p_1} K^h_{ijk} = \Omega_{p_1} K^h_{ijk}. \qquad \ldots(2.3)$$

The bi-recurrent Bochner curvature tensor can be expressed as

$$\nabla_{p_1} \nabla_{p_2} K^h_{ijk} = \Omega_{p_1 p_2} K^h_{ijk} \qquad \ldots(2.4)$$

where bi-recurrence tensor field is given by

$$\Omega_{p_1 p_2} = \nabla_{p_2} \Omega_{p_1} + \Omega_{p_1} \Omega_{p_2}. \qquad \ldots(2.5)$$

Further,

$$\nabla_{p_1} \nabla_{p_2} \nabla_{p_3} K^h_{ijk} = \Omega_{p_1 p_2 p_3} K^h_{ijk}$$

in which

$$\Omega_{p_1 p_2 p_3} = \nabla_{p_3} \Omega_{p_1 p_2} + \Omega_{p_1 p_2} \Omega_{p_3}. \qquad \ldots(2.6)$$

Proceeding in this way, we obtain the required result for Kaehler s-recurrent space $(s \geqslant 2)$.

Theorem 2.3 yields the following.

Theorem 2.4 — Every Kaehler s-recurrent space implies that it is Kaehler-$(s - 1)$ recurrent space, but the converse is not true in general.

Theorem 2.5 — The recurrence tensor field $\Omega_{p_1 p_2 \ldots p_s}$ is symmetric in the first pair of indices, if in Kaehler recurrent space of first order, the recurrence vector field is gradient, i.e.,

$$\nabla_{p_2} \Omega_{p_1} - \nabla_{p_1} \Omega_{p_2} = 0. \qquad \ldots(2.7)$$

Using eqns. (1.4) and (2.1), we get the following.

Theorem 2.6 — The Bianchi identity including s-recurrent tensor field can be expressed as

$$\Omega_{p_1 p_2 \ldots p_s} K_{khij} + \Omega_{kp_2 \ldots p_s} K_{hp_1 ij} + \Omega_{hp_2 \ldots p_s} K_{p_1 kij} = 0 \qquad \ldots(2.8)$$

in which the indices p_1, k and h have the cyclic rotation.

Definition 2.2 — If the holomorphically projective curvature tensor satisfies the relation

$$(\nabla_{p_1} \nabla_{p_2} \ldots \nabla_{p_s}) P^h_{ijk} = \tilde{\Omega}_{p_1 p_2 \ldots p_s} P^h_{ijk} \qquad \ldots(2.9)$$

$\tilde{\Omega}_{p_1 p_2 \ldots p_s} \neq 0$, then the Kaehler space K_n is said to be s-recurrent holomorphically projective Kaehler space. We shall call it Kaehlerian projective s-recurrent space.

In analogy to Theorem 2.2, we have the following.

Theorem 2.7 — If the Riemannian curvature tensor is s-recurrent then the holomorphically projective curvature tensor is also s-recurrent with the same recurrence tensor field.

PROOF : The relation

$$(\nabla_{p_1} \nabla_{p_2} \ldots \nabla_{p_s}) R^h_{ijk} = \tilde{\Omega}_{p_1 p_2 \ldots p_s} R^h_{ijk}, \qquad \ldots(2.10)$$

eqns. (1.3), (1.13) and (2.9) yield the required result.

Theorem 2.2, Theorem 2.7, eqns. (1.2), (1.3), (1.13), (2.1), (2.9) and (2.10) yield the following.

Theorem 2.8 — A Kaehlerian s-recurrent space is Kaehlerian projective s-recurrent space if the recurrence tensor fields are equal i.e.,

$$\Omega_{p_1 p_2 \ldots p_s} = \tilde{\Omega}_{p_1 p_2 \ldots p_s}. \qquad \ldots(2.11)$$

Theorem 2.9 — Every Kaehlerian projective s-recurrent space is a Kaehlerian s-recurrent space with Bochner curvature.

PROOF : Theorem 2.8, eqns. (1.13), (2.9), (1.2), (1.3), (1.6) and (1.8), after a simple calculation, will yield the required result.

3. Kaehlerian Spaces with Generalized Parallel Bochner Curvature Tensor

Definition 3.1 — We shall call K^*_n to be Kaehler space with generalized parallel Bochner curvature tensor if the following relation satisfies

$$\nabla_{p_1} \nabla_{p_2} \ldots \nabla_{p_s} K^h_{ijk} = 0. \qquad \ldots(3.1)$$

Comparing eqns. (2.1) and (3.1), we have the following.

Theorem 3.1 — Since $K^h_{ijk} \neq 0$, the recurrence tensor of s order, that is, $\Omega_{p_1 p_2} \ldots_{p_s}$ must be zero.

Theorem 3.2 — For Kaehler space with generalized parallel Bochner curvature tensor, it is necessary that the space should be Kaehler $(s-1)$-recurrent space, i.e.,

$$\Omega_{p_1 p_2 \cdots p_{s-1}} \neq 0. \qquad \ldots(3.2)$$

Theorem 3.3 — For K^*_n, the recurrence tensor of $(s-1)$ order must be recurrent of first order with a negative sign i.e.,

$$\nabla_{p_s} (\Omega_{p_1 p_2 \cdots p_{s-1}}) = (-\Omega_{p_s})\, \Omega_{p_1 p_2 \cdots p_{s-1}}. \qquad \ldots(3.3)$$

Definition 3.2 — Kaehler space, for which the holomorphically curvature tensor P^h_{ijk} satisfies

$$\nabla_{p_1} \nabla_{p_2} \ldots \nabla_{p_s} P^h_{ijk} = 0 \qquad \ldots(3.4)$$

is said to be Kaehlerian projective generalized symmetric space.

In analogy to the Theorem 2.9, Definitions 3.1 and 3.2 will illustrate the following theorem :

Theorem 3.4 — A necessary and sufficient condition for a Kaehler space to be Kaehler projective generalized symmetric space is that, it should be a Kaehler space with generalized parallel Bochner curvature tensor.

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