Indian J. pure appl. Math., 10(6) : 778-784, June 1979

ON THE COMPRESSIBLE FLOWS AROUND A CIRCULAR CYLINDER AND A SPHERE

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(*Received* 1 *July* 1978; *after revision* 22 *December* 1978)

In this paper various expansion velocities due to the flow caused by uniformly expanding circular cylinder and sphere in a perfect gas at rest have been discussed. Solutions are obtained by method of successive approximation in which first approximation gives the incompressible case and fluid velocity is calculated up to fourth approximation. Results are tabulated and shown graphically.

INTRODUCTION

The application of similarity flows in the supersonic and hypersonic steady flows is of practical importance. These type of flows have been studied analytically and numerically for the piston problem. Taylor (1946) discussed the flow around the expanding sphere. Sedov (1959) obtained general solution for the flow around the uniformly expanding circular cylinder and sphere. In the similarity flows of expanding circular cylinder and sphere in the gas, the velocity components are calculated by successive approximation by Kimura and Tsutahara (1977) and compared with Taylor's (1946) results.

In this paper the similarity flows caused by uniformly circular cylinder and sphere are given by the method of successive approximation. Howarth (1953), Van Dyke (1964) and Imai (1957) have given an out line for the approximation method with steady compressible flow when the flow field is irrotational and Mach number is less than unity. The accuracy of this method has been investigated by Sakurai (1975).

The fluid velocity has been expanded in terms of small parameter ϵ which is the ratio of kinetic energy to the total energy of fluid on the body surface; higher approximations are obtained by solving a set of differential equations with suitable boundary conditions on the body surface. It is clear from graphicals that the values of velocity increases slowly for cylinder and rapidly for sphere with the increase in ϵ.

BASIC EQUATIONS AND ANALYSIS

For compressible laminar flow over a circular cylinder and sphere the equations of continuity and momentum, in polar coordinates, are

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^j}\frac{\partial}{\partial r}(\rho v r^j) = 0 \qquad ...(1)$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad ...(2)$$

where the parameter $j = 0, 1$ and 2 for one, two and three dimensional flows i.e. flows caused by plane piston, a circular cylinder and sphere respectively. ρ, t, v- and p are density, time, fluid velocity and pressure. Introducing a non-dimensional variable r^* as

$$r^* = r/U_p t \qquad ...(3)$$

where U_p is the velocity of plane piston or the comparison velocity of circular cylinder or sphere, eqns. (1) and (2) are reduced to

$$(U_p r^* - v)\frac{1}{\rho}\frac{d\rho}{dr^*} = \frac{dv}{dr^*} + \frac{jv}{r^*} \qquad ...(4)$$

$$\frac{1}{\rho}\frac{dp}{dr^*} = (U_p r^* - v)\frac{dv}{dr^*}. \qquad ...(5)$$

If we take the flow as isentropic, ρ and p can be eliminated from (4) and (5) using sound speed $a = \left(\frac{dp}{d\rho}\right)^{1/2}$ to give

$$\frac{dv}{dr^*} + \frac{jv}{r^*} = \frac{1}{a^2}(U_p r^* - v)^2\frac{dv}{dr^*} \qquad ...(6)$$

Integration of eqn. (5) with respect to r^* from the body surface to any arbitrary point yields

$$\frac{a^2}{\gamma - 1} - \int_1^{r^*} U_p r^* \frac{dv}{dr^*}\, dr^* + \tfrac{1}{2}v^2 = \frac{a_p^2}{\gamma - 1} + \tfrac{1}{2}U_p^2 = \tfrac{1}{2}C^2 \qquad ...(7)$$

where a_p denotes the sound speed of the fluid on the body surface, γ is the ratio of specific heat at constant pressure and volume and C is a constant containing dimension of velocity. Eliminating a^2 from eqns. (6) and (7) and introducing a non-dimensional variable ϵ as

$$\epsilon = U_p^2/c^2 \qquad ...(8)$$

which is always less than unity, we have

$$(\gamma - 1)\left\{1 + 2\epsilon\left(\int_1^{r^*}\frac{r^*}{U_p}\frac{dv}{dr^*}\,dr^* - \tfrac{1}{2}\frac{V^2}{U_p^2}\right)\right\}\left(\frac{dv}{dr^*} + \frac{jv}{dr^*}\right)$$
$$= 2\epsilon\left(r^* - \frac{V}{U_p}\right)^2\frac{dV}{dr^*}. \qquad ...(9)$$

Solution of the Equation

Let us assume that fluid velocity may be expanded in the form

$$V = V_0 + \epsilon V_1 + \epsilon^2 V_2 + \epsilon^3 V_3 + \dots \qquad \dots(10)$$

where V_0 denotes the velocity of the fluid in the case of incompressible fluid, because the condition of incompressibility gives $\epsilon \to 0$ as $a \to \infty$.

Using boundary conditions on the body surface, eqns. (9) and (10) yield

$$V_0 = U_p;\ V_1, V_2, V_3, V_4, \dots = 1 \ \text{at}\ r^* = 1. \qquad \dots(11)$$

For one-dimensional flow

$$V_0 = U_p;\ V_1 = V_2 = V_3 = \dots = 0. \qquad \dots(12)$$

for two-dimensional flow

$$V_0 = \frac{U_p}{r^*} \qquad \dots(13)$$

$$V_1 = -\frac{U_p}{\gamma - 1}\left(r^* - 4\frac{\log r^*}{r^*} - \frac{1}{r^{*3}}\right) \qquad \dots(14)$$

$$\begin{aligned} V_2 = &-\frac{U_p}{(\gamma-1)^2}\left\{\frac{r^{*3}}{2} - 6r^* + \frac{2}{r^*} + \frac{2}{r^{*3}} - \frac{5}{2r^{*5}}\right. \\ &\left. + 4r^*\log r^* + \frac{20\log r^*}{r^*} - \frac{16(\log r^*)^2}{r^*} - \frac{12\log r^*}{r^{*3}}\right\} \\ &- \frac{U_p}{(\gamma-1)}\left\{-r^* + \frac{1}{2r^*} + \frac{1}{r^{*3}} - \frac{1}{2r^{*5}}\right. \\ &\left. + 2r^*\log r^* + \frac{2\log r^*}{r^*} - \frac{2\log r^*}{r^{*3}} - \frac{4(\log r^*)^2}{r^*}\right\} \end{aligned} \qquad \dots(15)$$

$$\begin{aligned} V_3 = &\frac{U_p}{(\gamma-1)^3}\left\{-36r^* + \frac{3}{2}r^{*2} + \frac{25}{8}r^{*3} - \frac{2}{5}r^{*5} + \frac{10}{r^{*3}}\right. \\ &- \frac{288}{r^{*2}} + \frac{56}{r^{*4}} + \frac{165}{8r^{*5}} - \frac{46}{5r^{*6}} + \frac{16}{3r^{*7}} \\ &+ 16\log r^* + \frac{10\log r^*}{r^*} - 208\frac{\log r^*}{r^{*2}} - 34\frac{\log r^*}{r^{*3}} - \frac{48\log r^*}{r^{*4}} \\ &+ 18r^*\log r^* - 4r^{*3}\log r^* + \frac{10(\log r^*)^2}{r^*} \\ &\left. - \frac{128}{r^{*2}}(\log r^*)^2 + \frac{48(\log r^*)^2}{r^{*3}} - \frac{16}{3}\frac{(\log r^*)^3}{r^*}\right\} \\ &+ \frac{2U_p}{(\gamma-1)^2}\left\{-\frac{27}{2}r^* - \frac{8}{9}r^{*2} + \frac{r^{*3}}{8} - \frac{34}{r^{*2}} - \frac{4}{r^{*3}} + \right. \end{aligned}$$

(*equation continued on p.* 781)

$$+\frac{217}{8r^{*5}}+\frac{5}{2r^{*7}}-\frac{14}{5r^{*6}}-8\log r^{*}+\frac{61}{2}r^{*}\log r^{*}$$

$$-\frac{4}{3}r^{*2}\log r^{*}+\frac{4}{3}\frac{\log r^{*}}{r^{*}}+\frac{24\log r^{*}}{r^{*2}}-\frac{51\log r^{*}}{r^{*3}}$$

$$-\frac{16}{r^{*4}}\log r^{*}+\frac{29}{2}\frac{\log r^{*}}{r^{*5}}-\frac{r^{*3}}{2}\log r^{*}-6r^{*}(\log r^{*})^{2}$$

$$-\frac{63}{2r^{*}}(\log r^{*})^{2}-\frac{16(\log r^{*})^{2}}{r^{*2}}+\frac{22(\log r^{*})^{2}}{r^{*3}}+\frac{38}{3r^{*}}(\log r^{*})^{3}\Big\}$$

$$+\frac{8U_p}{(\gamma-1)}\Big\{-\frac{r^{*}}{2}-\frac{1}{4r^{*3}}-\frac{1}{8r^{*5}}+\frac{1}{12r^{*7}}+r^{*}\log r^{*}$$

$$-\frac{\log r^{*}}{r^{*3}}+\frac{\log r^{*}}{2r^{*5}}-r^{*}(\log r^{*})^{2}-\frac{(\log r^{*})^{2}}{r^{*}}$$

$$+\frac{(\log r^{*})^{2}}{r^{*3}}+\frac{4}{3}\frac{(\log r^{*})^{3}}{r^{*}}\Big\} \qquad \ldots(16)$$

and for three-dimensional flow

$$V_0=\frac{U_p}{r^{*2}} \qquad \ldots(17)$$

$$V_1=-\frac{U_p}{(\gamma-1)}\Big\{2-\frac{9}{r^{*2}}+\frac{8}{r^{*3}}-\frac{1}{r^{*6}}\Big\} \qquad \ldots(18)$$

$$V_2=-\frac{U_p}{(\gamma-1)^2}\Big\{18-\frac{32}{r^{*}}-\frac{513}{10r^{*2}}+\frac{144}{r^{*3}}-\frac{78}{r^{*4}}$$

$$-\frac{27}{r^{*6}}+\frac{144}{5r^{*7}}-\frac{5}{2r^{*10}}\Big\}$$

$$-\frac{U_p}{(\gamma-1)}\Big\{8-\frac{16}{r^{*}}-\frac{63}{10r^{*2}}+\frac{32}{r^{*3}}-\frac{18}{r^{*4}}$$

$$-\frac{4}{r^{*6}}+\frac{24}{5r^{*7}}-\frac{1}{2r^{*10}}\Big\} \qquad \ldots(19)$$

$$V_3=\frac{8U_p}{(\gamma-1)^3}\Big\{-\frac{433}{40}-4r^{*}+\frac{11}{4}r^{*2}-\frac{r^{*4}}{3}+\frac{58}{r^{*}}$$

$$-\frac{67}{r^{*2}}\log r^{*}-\frac{36}{r^{*3}}-\frac{1741}{40r^{*4}}+\frac{3371}{60r^{*5}}$$

$$+\frac{1421}{40r^{*6}}-\frac{1987}{50r^{*7}}+\frac{305}{24r^{*8}}+\frac{249}{28r^{*9}}+\frac{157}{20r^{*10}}$$

$$+\frac{43}{15r^{*11}}-\frac{951}{200r^{*12}}-\frac{5}{22r^{*13}}+\frac{1}{2r^{*14}}+\frac{5}{26r^{*15}}\Big\}+$$

(equation continued on p. 782)

$$+\frac{8U_p}{(\gamma-1)^2}\left\{-\frac{79}{60}-\frac{26}{9}r^*+r^{*2}-\frac{1}{2r^*}-\frac{3\log r^*}{r^{*2}}\right.$$

$$-\frac{26}{r^{*3}}-\frac{179}{50r^{*4}}+\frac{56}{15r^{*5}}+\frac{4349}{240r^{*6}}$$

$$-\frac{863}{7r^{*7}}-\frac{531}{24r^{*8}}+\frac{212}{21r^{*9}}-\frac{113}{80r^{*10}}-\frac{10}{9r^{*11}}$$

$$\left.-\frac{143}{100r^{*12}}-\frac{1}{22r^{*13}}-\frac{17}{48r^{*14}}+\frac{1}{26r^{*15}}\right\}$$

$$+\frac{8U_p}{(\gamma-1)}\left\{4-\frac{8\log r^*}{r^{*2}}+\frac{8}{r^{*4}}-\frac{4}{3r^{*5}}-\frac{21}{4r^{*6}}\right.$$

$$\left.+\frac{23}{6r^{*8}}+\frac{8}{7r^{*9}}-\frac{4}{9r^{*11}}-\frac{1}{10r^{*12}}-\frac{1}{24r^{*14}}\right\}. \qquad ...(20)$$

It follows from these equations that fluid velocity for one-dimensional flow is equal to U_p and it is independent of ϵ. This corresponds to a well known flow in which a shock wave exists in front of the piston and fluid velocity head of the shock wave is at rest. For two and three-dimensional flows, the velocities V_0, V_1, V_2, V_3 are monotonic decreasing functions of r^* and when $r^* \to \infty$, V_1, V_2, V_3 tend to $-\infty$ for two-dimensional flow and approach finite values for three-dimensional flow. The same tendency is expected for the other higher terms. Since lowest fluid velocity V_0 approaches zero asymptotically as $r^* \to \infty$, we find that V decreases monotonically to a negative value as r^* increases when ϵ is non-zero.

RESULTS

The values of the fluid velocity for different cases are given in Tables I and II.

TABLE I

Fluid velocity V/U_p in case of circular cylinder

ϵ	r^*	V/U_p	
		up to ϵ^2	up to ϵ^3
0.00811	4.928	0.0971	0.0860
0.04773	1.992	0.3849	0.3047
0.16793	1.263	1.1184	0.8572
0.39398	1.085	4.2624	4.3406
0.46059	1.065	5.6169	5.7439
0.47244	1.062	5.9634	6.1016

TABLE II

Fluid velocity V/U_p in the case of sphere

ϵ	r^*	V/U_p up to ϵ^2	V/U_p up to ϵ^3
0.00811	4.928	0.00142	0.00111
0.04773	1.992	0.13041	0.10432
0.16793	1.263	0.57774	0.45858
0.39398	1.085	0.84188	0.8639
0.46059	1.065	0.87727	0.9252
0.47244	1.062	0.88265	0.9417

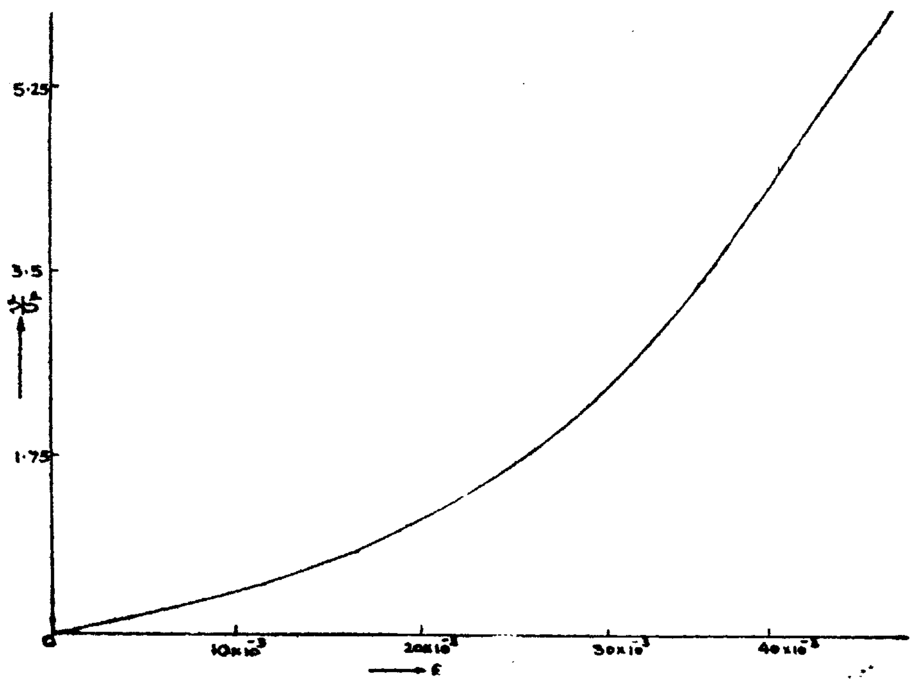

FIG. 1. Variations of V/U_p with respect to ϵ for circular cylinder up to 4th approximation with $\gamma = 1.405$.

Figures 1 and 2 show the variation of velocity for the flow past a circular cylinder and sphere.

Figure 1 represents the variation of velocity V/U_p with respect to ϵ. The velocity distribution is given up to 4th order approximation. It is clear from Fig. I that the

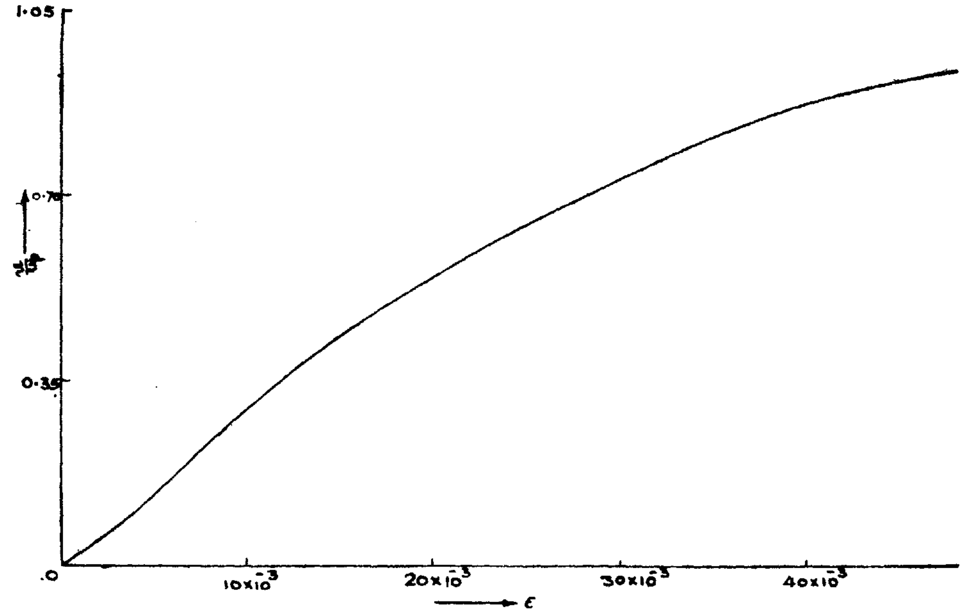

FIG. 2. Variations of V/U_p with respect to ϵ for large sphere up to 4th approximation with $\gamma = 1.405$.

velocity increases smoothly as ϵ increases. The variation is slow in the range $0 \leqslant \epsilon \leqslant 19 \times 10^{-3}$ and large in the range $19 \times 10^{-3} \leqslant \epsilon \leqslant 43 \times 10^{-3}$. For example the ratio of V/U_p for $\epsilon = 10 \times 10^{-3}$ to $\epsilon = 16 \times 10^{-3}$ is 1.96 and 2.01 for the range $\epsilon = 10 \times 10^{-3}$ to $\epsilon = 35 \times 10^{-3}$.

Fig. 2 shows that the variation of velocity is nearly constant and varies smoothly and continuously with increasing values of ϵ.

ACKNOWLEDGEMENT

The author is thankful to Prof. K. Lal for his guidance and to C.S.I.R., New Delhi, for the award of a scholarship. He is also thankful to the referee for valuable comments.

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