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# ON THE SPACE OF CERTAIN CLASS OF ANALYTIC FUNCTIONS

*by* P. D. SRIVASTAVA, *Department of Mathematics, Bipin Bihari College, Jhansi* 284002

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Let $(\mathcal{F}(\rho_0, T_0), d)$ denote the space of all functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$, analytic in the disc $|z| < R\,(0 < R < \infty)$, having growth $(\rho_0, T_0)$ with topology $d$ generated by the family of norms $\{\| f; \rho_0, T_0 + \delta \|; \delta > 0\}$ where

$$\| f; \rho_0; T_0 + \delta \| = |a_0| + \sum_{k=1}^{\infty} |a_k| R^k \exp(-AK^{\rho_0/(\rho_0+1)}(T_0 + \delta)^{1/(\rho_0+1)})$$

where $A = \dfrac{\rho_0 + 1}{\rho_0^{\rho_0/(\rho_0+1)}}$.

The present paper contains characterization of its continuous linear functionals, linear transformations of $\mathcal{F}(\rho_0, T_0)$ into itself and proper basis in terms of growth conditions on basis sequence $\{f_k\}$. A classical interpretation of convergence in $(\mathcal{F}(\rho_0, T_0), d)$ has been obtained.

## 1. INTRODUCTION

In the study of bases in the space $\Gamma$ of entire functions $\alpha = \alpha(z) = \sum_{0}^{\infty} a_n z^n$ topologized by the metric $|\alpha - \beta|$, where

$$|\alpha| = \max \{ |a_0|, |a_n|^{1/n}, n \geqslant 1\}. \qquad \ldots(1.1)$$

Iyer (1952) introduced the concept of proper basis and established a relationship between proper bases and automorphisms of the space. Arsove (1957) modified Iyer's definition of proper basis so as to make it possess more characteristic properties of the natural basis $\{z^n\}$, $n = 0, 1, 2, \ldots$ and obtained a characterization of the linearly homeomorphic images of the space into itself as those closed subspace admitting proper bases.

Later Krishnamurthy (1960) studied the space $\Gamma(\rho, d)$ of entire functions of order $\rho$ and type not exceeding '$d$' with metric topology defined by Iyer (1960).

Arsove (1958) also extended the basis theory to the space of functions analytic in the disc $|z| < R$, the topology being that of uniform convergence on compact sets and obtained the relationship between proper basis and linear homeomorphisms.

Recently, considerable interest has been shown by different workers (Maclane 1963, Kapoor 1972) in studying the growth of functions analytic in the disc $D = \{z : |z| < R\}$. Analogous to the study of entire functions, the concepts of order and type have been introduced. Thus $f(z) = \sum_{0}^{\infty} a_k z^k$, analytic in

$$|z| < R\,(0 < R < \infty)$$

is said to be of order $\rho_0$ if

$$\limsup_{r\to R} \frac{\log^+ \log^+ M(r; f)}{-\log\log \frac{R}{r}} = \rho_0. \qquad \ldots(1.2)$$

If $0 < \rho_0 < \infty$, $f(z)$ is said to be of type $T_0$ if

$$\limsup_{r\to R} \frac{\log M(r; f)}{\left(\log \frac{R}{r}\right)^{-\rho_0}} = T_0. \qquad \ldots(1.3)$$

The coefficient characterization for $\rho_0$ and $T_0$ has also been obtained (Maclane 1963, Kapoor 1972). In fact, it is shown that

$$\rho_0/(\rho_0+1) = \limsup_{k\to\infty} \frac{\log^+ \log^+ |a_k| R^k}{\log k} \qquad \ldots(1.4)$$

$$\limsup_{k\to\infty} \frac{(\log^+ |a_k| R^k)^{\rho_0+1}}{k^{\rho_0}} = T_0 A^{\rho_0+1} \qquad \ldots(1.5)$$

where $A = \dfrac{(\rho_0+1)}{\rho_0^{\rho_0/(\rho_0+1)}}$ and $\log^+ x = \max(\log x, 0)$.

Let $U_R(\rho_0, T_0)$ denote the class of all functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$, analytic in the disc $|z| < R\,(0 < R < \infty)$, whose order does not exceed $\rho_0$ and whose type does not exceed $T_0$ if of order $\rho_0$. It is easily seen that $U_R(\rho_0, T_0)$ is a linear space over the complex field $C$ with the usual addition and scalar multiplication. Further, any element $f(z) = \sum_{0}^{\infty} a_k z^k \in U_R(\rho_0, T_0)$ is characterized by the equation

$$\limsup_{k\to\infty} k^{-\rho_0} (\log^+ |a_k| R^k)^{\rho_0+1} \leqslant A^{\rho_0+1} T_0. \qquad \ldots(1.6)$$

Define

$$\|f; \rho_0, T_0+\delta\| = |a_0| + \sum_{k=1}^{\infty} |a_k| R^k P(k, \rho_0, T_0+\delta) \qquad \ldots(1.7)$$

where $$P(k, \rho_0, T_0 + \delta) = \exp\left(-Ak^{Q}(T_0 + \delta)^{1/(\rho_0+1)}\right)$$
$$Q = \rho_0/(\rho_0 + 1).$$

Clearly, for every $\delta > 0$ and $f \in U_R(\rho_0, T_0)$, (1.7) defines a norm. Denote the corresponding normed space by $\mathcal{F}(\rho_0, T_0, \delta)$ and let $\mathcal{F}(\rho_0, T_0)$ be the weakest topology which is stronger than each $\mathcal{F}(\rho_0, T_0, \delta)$. Obviously, $\mathcal{F}(\rho_0, T_0)$ is generated by the family $\{\mathcal{F}(\rho_0, T_0, \delta), \delta > 0\}$. Further, it can be easily verified that $\mathcal{F}(\rho_0, T_0)$ is an $F$-space (Dunford and Schwartz 1958) under the induced metric

$$d(f, g) \equiv |||f - g||| = \sum_{p=1}^{\infty} \frac{1}{2^p} \frac{\|f - g; \rho_0; T_0 + 1/p\|}{1 + \|f - g; \rho_0; T_0 + 1/p\|}. \quad \ldots(1.8)$$

In this paper, we study different properties of the space $\mathcal{F}(\rho_0, T_0)$ such as dual, linear transformations of $\mathcal{F}(\rho_0, T_0)$ into itself, classical interpretation of convergence in $\mathcal{F}(\rho_0, T_0)$ etc. We also give the characterisation of proper basis in terms of growth conditions on $\{f_k\}$.

## 2. Preliminary Lemmas

In this section we state a few Lemmas which are either well known or can be easily proved on the lines adopted by Krishnamurthy (1960).

*Lemma* 1 — The set $\mathcal{F}^*(\rho_0, T_0)$ of continuous linear functionals on $\mathcal{F}(\rho_0, T_0)$ is the union of the sets $\mathcal{F}^*(\rho_0, T_0, \delta)$ for all $\delta > 0$.

*Lemma* 2 — Let $L$ be a linear transformation of $\mathcal{F}(\rho_0, T_0)$ into itself. In order that $L$ is continuous, it is necessary and sufficient that, for each $\delta_2 > 0$, there exists some $\delta_1 > 0$ such that $L$ is continuous linear transformation from $\mathcal{F}(\rho_0, T_0, \delta_1)$ into $\mathcal{F}(\rho_0, T_0, \delta_2)$.

*Lemma* 3 [Dunford and Schwartz (1958), Theorem 5, p. 58] — If a linear space $V$ is an $F$-space under each of two metrics, and if one of the corresponding topologies contains the other, the two topologies are equal.

*Lemma* 4 — If $|||f||| \geqslant k$ $(0 < k < 2)$, then $\|f; \rho_0, T_0 + \delta\| \geqslant k/(2 - k)$ for some $\delta = \delta_0$ where $0 < \delta_0 \leqslant 1$, and therefore for all values of $\delta \leqslant \delta_0$.

*Remark* : A consequence of this lemma is that if a series converges in $\mathcal{F}(\rho_0, T_0, \delta)$ for each $\delta > 0$, then it converges in $\mathcal{F}(\rho_0, T_0)$.

## 3. Dual Space $\mathcal{F}^*(\rho_0, T_0)$

In this section we obtain the general form of continuous linear functionals on $\mathcal{F}(\rho_0, T_0)$. Thus we have

*Theorem* 1 — (a) For each $\delta > 0$, $\psi \in \mathcal{F}^*(\rho_0, T_0, \delta)$ if and only if $\psi(f) = \sum_{k=0}^{\infty} c_k a_k$ for all $f(z) = \sum_0^{\infty} a_k z^k \in \mathcal{F}(\rho_0, T_0, \delta)$, where $\{c_k\}_{k=0}$ is a sequence in $C$ such that

$$| c_k | R^{-k}/P(k, \rho_0, T_0 + \delta) \text{ is bounded.} \qquad \ldots(3.1)$$

(b) $\psi \in \mathcal{F}^*(\rho_0, T_0)$ if and only if $\psi(f) = \sum_0^{\infty} c_k a_k$ for all

$$f(z) = \sum_0^{\infty} a_k z^k \in \mathcal{F}(\rho_0, T_0)$$

where $\{c_k\}_{k=0}^{\infty}$ is a sequence in $C$ such that

$$\limsup_{k \to \infty} ( | c_k | R^{-k})^{k^{-Q}} < \exp(- A T_0^{1/(\rho_0+1)}). \qquad \ldots(3.2)$$

PROOF : (a) Let $\delta_k = z^k$ and $\psi(\delta_k) = c_k$ $(k \geqslant 0)$ and suppose $\psi \in \mathcal{F}^*(\rho_0, T_0, \delta)$, then

$$\psi(f) = \lim_{k \to \infty} (c_0 a_0 + \ldots + c_k a_k) = \sum_0^{\infty} c_k a_k.$$

Since $\psi$ is continuous, so given $\delta > 0$ there exists a constant $M(\delta)$ such that

$$| \psi(f) | \leqslant M \| f; \rho_0, T_0 + \delta \|.$$

Putting $f = z^k$, we get

$$| c_k | \leqslant M \| z^k; \rho_0; T_0 + \delta \| = M R^k P(k, \rho_0, T_0 + \delta)$$

which yields (3.1).

Conversely, let $\psi(f) = \sum_0^{\infty} c_k a_k$, where $\{c_k\}$ satisfies (3.1). Obviously $\psi$ is linear. To show $\psi$ is continuous, consider

$$| \psi(f) | = | \sum_0^{\infty} c_k a_k | \leqslant \sum_0^{\infty} M R^k P(k, \rho_0, T_0 + \delta) | a_k |$$

$$= M \| f; \rho_0, T_0 + \delta \|$$

so that $f \in \mathcal{F}^*(\rho_0, T_0, \delta)$.

The result in Theorem 1(b) immediately follows in view of the above result and Lemma 1.

## 4. Convergence in $\mathcal{F}(\rho_0, T_0)$

In this section we obtain the classical interpretation of convergence in $\mathcal{F}(\rho_0, T_0)$. Thus we have

*Theorem* 2 — Let $\{f_k\}$ be a sequence of elements of $\mathcal{F}(\rho_0, T_0)$. The statement $f_k \to f$ in $\mathcal{F}(\rho_0, T_0)$ is equivalent to the statement that for every $\delta > 0$, the sequence $f_k(z)$ of analytic functions converges to $f(z)$ uniformly over the disk $| z | < R$ relative to the function $\exp \left\{(T_0 + \delta)\left(\log \frac{R}{|z|}\right)^{-\rho_0}\right\}$.

Proof: Define

$$\| f; \rho_0; T_0 + \delta \|_1 = \max_z \left\{\exp\left(-(T_0 + \delta)\left(\log \frac{R}{|z|}\right)^{-\rho_0}\right\} | f(z) | \quad \ldots(4.1)$$

where $z$ varies on the disc $= \{z : | z | = r < R\}$.

Clearly, for every $\delta > 0$ and $f \in U_R(\rho_0, T_0)$, (4.1) defines a norm. Denote the corresponding normed space by $\mathcal{F}'(\rho_0, T_0, \delta)$. Let $\mathcal{F}'(\rho_0, T_0)$ be the topology generated by the family $\{\mathcal{F}'(\rho_0, T_0, \delta) : \delta > 0\}$. Obviously $\mathcal{F}'(\rho_0, T_0)$ is an $F$-space (Dunford and Schwartz 1958) under the invariant metric

$$d'(f, g) = ||| f - g |||_1 = \sum_{p=1}^{\infty} \frac{1}{2^p} \frac{\| f - g; \rho_0; T_0 + 1/p \|_1}{1 + \| f - g; \rho_0; T_0 + 1/p \|_1}. \quad \ldots(4.2)$$

We now show that the topology $\mathcal{F}'(\rho_0, T_0)$ is comparable with $\mathcal{F}(\rho_0, T_0)$. For this, consider

$$\max_{\substack{z \\ 0<|z|<R}} | a_k | r^k \exp\left(-(T_0 + \delta)\left(\log \frac{R}{|z|}\right)^{-\rho_0}\right)$$

$$= | a_k | \max_r \exp\left(-(T_0 + \delta)\left(\log \frac{R}{r}\right)^{-\rho_0} + k \log r\right).$$

Thus; at $r = R \exp\left(-\left(\frac{\rho_0(T_0 + \delta)}{k}\right)^{1/(\rho_0+1)}\right)$ the above expression has maximum value. So

$$= | a_k | \exp\left\{-(T_0 + \delta)\left(\log R - \log R + \left(\frac{\rho_0(T_0 + \delta)}{k}\right)^{1/(\rho_0+1)}\right)^{\rho-0}\right.$$

$$\left. + k \log R - K\left(\frac{\rho_0(T_0 + \delta)}{k}\right)^{1/(\rho_0+1)}\right\}$$

*(equation continued on p. 89)*

$$= | a_k | R^k \exp(-(T_0+\delta)^{1/(\rho_0+1)} K^{\rho_0/(\rho_0+1)} \rho_0^{-\rho_0/(\rho_0+1)}(1+\rho_0))$$

$$= | a_k | R^k \exp(-AK^Q(T_0+\delta)^{1/(\rho_0+1)}).$$

Hence the topologies are comparable. Thus Lemma 3 gives that metrics (1.8) and (4.2) are equal. This completes the proof.

*Theorem* 3 — A necessary and sufficient condition that there exists a continuous linear transformation $L$ from $\mathcal{F}(\rho_0, T_0)$ into itself with $L(e_k)=f_k$, $k=0,1,2,\ldots, e_k \equiv z^k$, is that, for each $\delta > 0$

$$\limsup_{k\to\infty} (\| f_k;\rho_0;\, T_0+\delta \| R^{-k})^{k^{-Q}} < \exp(-AT_0^{1/(\rho_0+1)}). \qquad \ldots(4.3)$$

PROOF : Suppose $L$ is a continuous linear transformation from $\mathcal{F}(\rho_0, T_0)$ into itself. Then, by Lemma 2, for every $\delta > 0$, there exists a $\delta_1(\delta) > 0$ such that $L$ is continuous linear transformation from $\mathcal{F}(\rho_0, T_0, \delta_1)$ into $\mathcal{F}(\rho_0, T_0, \delta)$. Hence there exists a constant $M(\delta)$ such that

$$\| L(z^k);\, \rho_0;\, T_0+\delta \| \leqslant M \| z^k;\, \rho_0;\, T_0+\delta_1 \|$$

i.e. $$\| f_k;\, \rho_0;\, T_0+\delta \| \leqslant MR^k \exp(-Ak^Q(T_0+\delta_1)^{1/(\rho_0+1)})$$

which gives (4.3).

Conversely, let sequence $\{f_k\}$ satisfy (4.3). So given $\eta' > 0$, there exists $k_0'(\eta')$ such that for $k \geqslant k_0(\eta')$

$$(R^{-k} \| f_k:\, \rho_0;\, T_0+\delta \|)^{k^{-Q}} \leqslant \exp(-A(T_0+\eta')^{1/(\rho_0+1)}). \qquad \ldots(4.4)$$

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{F}(\rho_0, T_0)$. Take $\eta\ (0<\eta<\eta')$. From (1.6) there exists $k_0(\eta)$ such that for $k \geqslant k_0(\eta)$

$$| a_k | \leqslant R^{-k} \exp(Ak^Q(T_0+\eta)^{1/(\rho_0+1)}). \qquad \ldots(4.5)$$

Let $k' = \max(k_0, k_0')$, then (4.4) and (4.5) lead to

$$| a_k | \| f_k;\, \rho_0;\, T_0+\delta \| \leqslant \exp\{Ak^Q(T_0+\eta)^{1/(\rho_0+1)} - Ak^Q(T_0+\eta')^{1/(\rho_0+1)}\}$$
$$\text{for } k \geqslant k',$$

which gives that the series $\Sigma\, | a_k | \, \| f_k;\, \rho_0;\, T_0+\delta \|$ is convergent for every $\delta > 0$. Hence, by Lemma 4, series $\sum_0^{\infty} a_k f_k(z)$ converges to an element in $\mathcal{F}(\rho_0, T_0)$. Define $L(f) = \sum_0^{\infty} a_k f_k$, for $f \in \mathcal{F}(\rho_0, T_0)$. Then $L(e_k) = f_k$. Given $\delta > 0$, $\delta' > 0$, we have, from (4.3), for $k \geqslant k_0''(\delta, \delta_1)$,

$$\| f_k ; \rho_0 ; T_0 + \delta \| \leqslant M_1 R^k \exp\left(- A k^Q (T_0 + \delta')^{1/(\rho_0+1)}\right) \text{ for every } k.$$

Thus

$$\| L(f) ; \rho_0 ; T_0 + \delta \| \leqslant \Sigma \mid a_k \mid \| f_k ; \rho_0 ; T_0 + \delta \|$$

$$= M_1 \sum_{0}^{\infty} \mid a_k \mid R^k \exp\left(- A k^Q (T_0 + \delta')^{1/(\rho_0+1)}\right)$$

$$= M_1 \| f ; \rho_0 ; T_0 + \delta' \|$$

which shows that $T$ is continuous linear transformation from $\mathcal{F}(\rho_0, T_0, \delta_1)$ into $\mathcal{F}(\rho_0, T_0, \delta)$ for every $\delta > 0$. Therefore by Lemma 2, $T$ is continuous linear transformation from $\mathcal{F}(\rho_0, T_0)$ into itself.

## 5. Proper Bases in $\mathcal{F}(\rho_0, T_0)$ and their Characterization

Let $\{f_k\}$, $k = 0, 1, 2, \ldots$ be a sequence of functions in $U_R(\rho_0, T_0)$. If

$$\sum_{k=0}^{\infty} c_k f_k = 0 \Rightarrow c_k = 0$$

for all $\{c_k\}$ of complex numbers for which $\Sigma\, c_k f_k$ converges in $\mathcal{F}(\rho_0, T_0)$, the sequence $\{f_k\}$ will be called L.I. We shall say that $\{f_k\}$ spans a subspace $\mathcal{F}_0(\rho_0, T_0)$ of $\mathcal{F}(\rho_0, T_0)$ provided $\mathcal{F}_0(\rho_0, T_0)$ consists of all linear combination $\sum_{0} c_k f_k$ where $\{c_k\}$ is any sequence of complex numbers for which the series converges in $\mathcal{F}(\rho_0, T_0)$. A sequence $\{f_k\}$ which is L.I. and spans a closed subspace $\mathcal{F}_0(\rho_0, T_0)$ of $\mathcal{F}(\rho_0, T_0)$ is called a basis in $\mathcal{F}_0(\rho_0, T_0)$.

A basis $\{f_k\}$ in a subspace $\mathcal{F}_0$ of $\mathcal{F}(\rho_0, T_0)$ is said to be proper if for all sequences $\{c_k\}$ of complex numbers,

$$\sum_{0}^{\infty} c_k f_k \text{ converges in } \mathcal{F}(\rho_0, T_0) \text{ if and only if}$$

$$\sum_{0}^{\infty} c_k e_k \text{ converges in } \mathcal{F}(\rho_0, T_0). \qquad \ldots(5.1)$$

Now, it can be easily seen that

$$\sum_{0}^{\infty} c_k e_k \text{ converges in } \mathcal{F}(\rho_0, T_0) \text{ if and only if}$$

$$\limsup_{k\to\infty} k^{-\rho_0} (\log^+ \mid c_k \mid R^k)^{\rho_0+1} \leqslant A^{\rho_0+1}\, T_0. \qquad \ldots(5.2)$$

We first have the following :

*Theorem* 4 — The following three conditions are equivalent.

(A) $\limsup_{k\to\infty} (\| f_k; \rho_0; T_0 + \delta \| R^{-k})^{k^Q} < \exp(-AT_0^{1/(\rho_0+1)})$
for every $\delta > 0$.

(B) For all sequences $\{c_k\}$ of complex numbers, $\sum_0^\infty c_k e_k$ converges in $\mathcal{F}(\rho_0, T_0) \Rightarrow \sum_0^\infty c_k f_k$ converges in $\mathcal{F}(\rho_0, T_0)$.

(C) For all sequences $\{c_k\}$ of complex numbers, $\sum_0^\infty c_k e_k$ converges in $\mathcal{F}(\rho_0, T_0) \Rightarrow c_k f_k \to 0$ in $\mathcal{F}(\rho_0, T_0)$.

PROOF: Clearly, (B) $\Rightarrow$ (C). (A) $\Rightarrow$ (B) has been proved in sufficient part of Theorem 3. So we have only to show (C) $\Rightarrow$ (A). Assume (C) holds and (A) does not. So, given $\delta' > 0$, there exists a sequence $\{k_n\}$ of positive integers such that

$$(\| f_k; \rho_0; T_0 + \delta \| R^{-k})^{k^{-Q}} \geqslant \exp\left(-A\left(T_0 + \frac{1}{n}\right)^{1/(\rho_0+1)}\right) \text{for } k = k_n. \quad \ldots(5.3)$$

Define

$$c_k = \begin{cases} \| f_k; \rho_0; T_0 + \delta' \|^{-1} & \text{when } k = k_n \\ 0 & \text{when } k \neq k_n. \end{cases} \quad \ldots(5.4)$$

Clearly $\{c_k\}$ satisfies (5.2). So $\Sigma\, c_k e_k$ converges in $\mathcal{F}(\rho_0, T_0)$. Thus (C) gives $c_k f_k \to 0$ in $\mathcal{F}(\rho_0, T_0)$. But for all $k = k_n$,

$$\| c_k f_k; \rho_0; T_0 + \delta' \| = | c_{k_n} | \, \| f_{k_{n'}}; \rho_0; T_0 + \delta' \| = 1$$

gives the contradiction to $c_k f_k \to 0$ in $\mathcal{F}(\rho_0, T_0)$.
This proves (C) $\Rightarrow$ (A).

*Theorem* 5 — The following three conditions are equivalent :

($\alpha$) $\lim_{\delta\to 0} \{\liminf_{k\to\infty} (R^{-k} \| f_k; \rho_0; T_0 + \delta \|)^{k^{-Q}}\} \geqslant \exp(-AT_0^{1/(\rho_0+1)})$

($\beta$) For all sequences $\{c_k\}$ of complex numbers, $\Sigma\, c_k f_k$ converges in $\mathcal{F}(\rho_0, T_0) \Rightarrow \Sigma\, c_k e_k$ converges in $\mathcal{F}(\rho_0, T_0)$.

($\gamma$) For all sequences $\{c_k\}$ of complex numbers, $c_k f_k \to 0$ in $\mathcal{F}(\rho_0, T_0) \Rightarrow \Sigma\, c_k e_k$ converges in $\mathcal{F}(\rho_0, T_0)$.

PROOF: It is clear that ($\gamma$) $\Rightarrow$ ($\beta$). We shall prove ($\beta$) $\Rightarrow$ ($\alpha$) and ($\alpha$) $\Rightarrow$ ($\gamma$). To prove ($\beta$) $\Rightarrow$ ($\alpha$), suppose ($\beta$) holds and ($\alpha$) does not. Then

$$\lim_{\delta\to 0} \{\liminf_{k\to\infty} (R^{-k} \| f_k; \rho_0; T_0 + \delta \|)^{k^{-Q}}\} < \exp(-AT_0^{1/(\rho_0+1)}). \quad \ldots(5.5)$$

so that

$$\liminf_{k\to\infty} (R^{-k} \| f_k; \rho_0; T_0 + \delta \|)^{k^{-Q}} < \exp(-AT_0^{1/(\rho_0+1)}) \text{ for each } \delta > 0. \quad \text{...(5.6)}$$

Fix $\eta > 0$. In view of (5.5) and (5.6), we can find, for each $r > 0$, a positive integer $k_r$ such that, for all $r$, $k_{r+1} > k_r$, and for $k = k_r$

$$(R^{-k} \| f_k; \rho_0; T_0 + (1/r) \|)^{k^{-Q}} < \exp(-A(T_0 + \eta)^{1/(\rho_0+1)}). \quad \text{...(5.7)}$$

Let $0 < \eta_1 < \eta$ and define a sequence $\{c_k\}$ by

$$c_k = \begin{cases} R^{-k} \exp(Ak^Q(T_0 + \eta_1)^{1/(\rho_0+1)}) & \text{when } k = k_r \\ 0 & \text{when } k \neq k_r. \end{cases} \quad \text{...(5.8)}$$

Then, for any $\delta > 0$,

$$\sum_{k=0}^{\infty} | c_k | \| f_k; \rho_0; T_0 + \delta \| = \sum_{r=1}^{\infty} | c_{k_r} | \| f_{k_r}; \rho_0; T_0 + \delta \|$$

which is dominated by $\Sigma | c_{k_r} | \| f_{k_r}; \rho_0; T_0 + (1/r) \|$ after omitting finitely many terms for which $\frac{1}{r} > \delta$ (given). The convergence of this series follows from (5.7) and (5.8). Thus the sequence $\{c_k\}$ has the property that $\Sigma c_k f_k$ converges in $\mathcal{F}(\rho_0, T_0, \delta)$ for each $\delta > 0$ and therefore in $\mathcal{F}(\rho_0, T_0)$. Because of ($\beta$), $\Sigma c_k e_k$ converges in $\mathcal{F}(\rho_0, T_0)$ so that (5.2) is satisfied. But by (5.8),

$$K_r^{-Q} \log^+(| c_{k_r} | R^{k_r}) = A(T_0 + \eta_1)^{1/(\rho_0+1)} > AT_0^{1/(\rho_0+1)}$$

so
$$\limsup_{k\to\infty} k^{-\rho_0} (\log^+ | c_k | R^k)^{\rho_0+1} > A^{\rho_0+1} T_0$$

which contradicts (5.2). Thus ($\beta$) $\Rightarrow$ ($\alpha$).

To show ($\alpha$) $\Rightarrow$ ($\gamma$) assume that ($\alpha$) holds but ($\gamma$) does not. So there exists a sequence $\{c_k'\}$ for which

$$c_k' f_k \to 0 \text{ in } \mathcal{F}(\rho_0, T_0) \quad \text{...(5.9)}$$

but

$$\limsup_{k\to\infty} k^{-\rho_0} (\log^+ | c_k' | R^k)^{\rho_0+1} > A^{\rho_0+1} T_0.$$

Thus given $\lambda > 0$, there exists a sequence $\{k_n\}$ of positive integers such that

$$K^{-\rho_0} (\log^+ | c_k' | R^k)^{\rho_0+1} \geqslant A^{\rho_0+1}(T_0 + \lambda) \text{ for } k = k_n, n = 1, 2, \ldots . \quad \text{...(5.10)}$$

Choose a positive number $\eta$ such that $\lambda > \frac{3\eta}{2}$. From ($\alpha$), there exists a $\delta = \delta(\eta)$ such that

$$\liminf_{k \to \infty} (R^{-k} \| f_k; \rho_0; T_0 + \delta \|)^{k^{-Q}} \geqslant \exp(-A(T_0 + \eta)^{1/(\rho_0+1)})$$

so that

$$R^{-k} \| f_k; \rho_0; T_0 + \delta \| \geqslant \exp\left(-Ak^Q\left(T_0 + \frac{3\eta}{2}\right)^{1/(\rho_0+1)}\right) \text{for } k \geqslant k_0(\eta). \quad \ldots(5.11)$$

Thus

$$\max_k \| c'_k f_k; \rho_0; T_0 + \delta \| \geqslant \max_{k_n} | c'_{k_n} | \| f_{k_n}; \rho_0; T_0 + \delta \| > 1$$

from (5.10) and (5.11). It follows that for this $\delta$, $c'_k f_k \not\to 0$ in $\mathcal{F}(\rho_0, T_0, \delta)$. So $c'_k f_k \not\to 0$ in $\mathcal{F}(\rho_0, T_0)$ which is contradiction to (5.9). This proves ($\alpha$) $\Rightarrow$ ($\gamma$). Hence the theorem.

Combining Theorems 4 and 5, we get

*Theorem* 6 — A basis $\{f_k\}$ in a closed subspace $\mathcal{F}_0$ of $\mathcal{F}(\rho_0, T_0)$ is proper if and only if conditions (A) and ($\alpha$) hold.

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