Indian J. pure appl. Math., **10**(1) : 94-99, January 1979

RELATIVISTIC ELECTROMAGNETIC FLUIDS AND RICCI COLLINEATIONS

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(*Received* 18 *May* 1978)

This paper investigates Ricci collineations admitted by the electromagnetic fluid space-times with respect to the vorticity vector, electric field vector and magnetic field vector respectively. Further, this paper demonstrates a few examples of conformal motions which degenerate into motions in the space-time of electromagnetic fluids.

1. INTRODUCTION

Glass (1975) has investigated an interesting conservation expression which is generalized by Oliver and Davis (1976) employing symmetry methods in the domain of shear-free perfect fluids. An elegant account of groups of motions and Ricci collineations in the space-time filled with perfect magnetofluids is due to Shaha (1974) who has developed a definite magnetofluid scheme characterized by a stress-energy-momentum tensor with time like eigenvalue as the energy density and the three space-like eigenvalues as the partial pressures. For such scheme he has obtained a set of necessary and sufficient conditions for the Ricci collineations with respect to the world line to imply the motion. Asgekar and Date (1977) have investigated the Ricci collineations and conformal motions with respect to the flow vector in the space-time filled with imperfect magnetofluids. The author (Prasad 1978a–c, Prasad and Sinha 1978) has studied the family of contracted Ricci collineations admitted by the magnetofluid space-times and investigated Ricci collineations admitted by the electromagnetic fluid space-times with respect to the fluid flow vector.

The purpose of this paper is to explore certain theorems in the area of local conservation laws involving symmetry methods developed by Oliver and Davis (1976), Davis (1974) and Davis *et al.* (1976). In particular, we look at the symmetry properties in terms of $\underset{\xi}{\mathcal{L}} R_{ij} = 0$ for space-like symmetry vectors ξ^i and investigate some of the conditions that the Ricci collineations (RC) impose on the electromagnetic fluids.

2. FIELD EQUATIONS AND KINEMATICAL PARAMETERS

The Maxwell field equations read as

$$(u^i B^j - u^j B^i + \eta^{ijkl} u_k e_l)_{;j} = 0 \qquad \ldots(2.1)$$

and

$$(u^i D^j - u^j D^i + \eta^{ijkl} u_k h_l)_{;j} = - J^i \quad \text{...(2.2)}$$

where B^i is the magnetic induction vector, D^i the electric induction vector, J^i the electric current vector, e^i the electric field vector and h^i the magnetic field vector.

Einstein field equations are

$$R_{ij} - \tfrac{1}{2} R g_{ij} = - T_{ij} \quad \text{...(2.3)}$$

where the stress-energy-momentum tensor T_{ij} for a self-gravitating, thermally conducting, viscous, compressible and charged fluid with constant magnetic permeability and electric permittivity is given by

$$T_{ij} = (\rho^* + p^*) u_i u_j - p^* g_{ij} + \nu \sigma_{ij}$$
$$- (\lambda e_i e_j + \mu h_i h_j) + P_i u_j + P_j u_i \quad \text{...(2.4)}$$

where

$$\rho^* = \rho + \tfrac{1}{2}(\lambda \mid e \mid^2 + \mu \mid h \mid^2) \quad \text{...(2.5)}$$

$$p^* = p + \tfrac{1}{2}(\lambda \mid e \mid^2 + \mu \mid h \mid^2) \quad \text{...(2.6)}$$

$$P_i = q_i - V_i. \quad \text{...(2.7)}$$

Here ρ is the matter energy density of the fluid, p the isotropic pressure, $\nu (\geqslant 0)$ the coefficient of viscosity, q^i the heat energy-flux vector and V^i the electromagnetic energy-flux vector.

The kinematical properties of the fluid streamlines are characterized by the usual decomposition for the rate of change of the flow vector u^i (Ehlers 1961).

$$u_{i;j} = \sigma_{ij} + \omega_{ij} + \theta \gamma_{ij} + D u_i u_j \quad \text{...(2.8)}$$

where σ_{ij}, ω_{ij}, θ denote shear, rotation and expansion of the congruence of stream lines respectively. D stands for the directional derivative along the fluid flow.

The covariant derivative of the 4-vector n^i tangential to the space-like congruence is decomposed according to Greenberg (1970) as follows:

$$n_{i;j} = \overset{*}{\sigma}_{ij} + \overset{*}{\omega}_{ij} + \overset{*}{\theta} \overset{*}{\gamma}_{ij} - D^* n_i n_j - (D n_k u^k) u_i u_j$$
$$+ (D^* n_k u^k) u_i n_j + n_{k;j} u^k u_i \quad \text{...(2.9)}$$

where $\overset{*}{\sigma}_{ij}$, $\overset{*}{\omega}_{ij}$, $\overset{*}{\theta}$ denote the shear, rotation and expansion of the congruence formed by the magnetic field lines respectively. D^* stands for the directional derivative along the magnetic field lines and the projection tensor $\overset{*}{\gamma}_{ij}$ is defined as

$$\overset{*}{\gamma}_{ij} = g_{ij} - u_i u_j + n_i n_j. \quad \text{...(2.10)}$$

3. Electromagnetic Fluids and RC

In this section, we begin to investigate the relations between certain conditions on the electromagnetic fluids and the Ricci collineations (RC) admitted by the electromagnetic fluid space-times.

By virtue of (2.3) and (2.4), we get

$$R_{ij} = - \{\mu_0 u_i u_j - \mu_1 \gamma_{ij} + \nu \sigma_{ij} - \lambda e_i e_j - \mu h_i h_j + P_i u_j + P_j u_i\} \quad ...(3.1)$$

where

$$2\mu_0 = \rho + 3p + \mu \mid h \mid^2 + \lambda \mid e \mid^2 ; \; 2\mu_1 = \rho - p + \mu \mid h \mid^2 + \lambda \mid e \mid^2 .$$

Let us consider the space-like symmetry mapping vectors of the form $\xi^i = \varphi \omega^i$, i.e. symmetry mapping along the direction of the fluid vorticity then we observe the following theorem:

Theorem 3.1 — For an electromagnetic fluid with $\sigma_{ij}\omega^i\omega^j = 0$ admitting the symmetry property $\underset{\xi}{\mathcal{L}} R_{ij} = 0$ for $\xi^i = \varphi\omega^i$, the conservation law $(\alpha\mu_0\omega^k)_{;k} = 0$, where $\alpha_{,k}\omega^k = 0$ holds if (i) the 'energy-flux' vector P^i is parallel or antiparallel to the axis of rotation of the fluid and (ii) $D\omega^2 = 2\{D \ln \varphi - 11\theta\}$.

Proof : It can be shown that $u^i u^j \underset{\xi}{\mathcal{L}} R_{ij} = 0$ is equivalent to

$$\mu_{0,k}\xi^k - 2\mu_0 \xi^k D u_k + 2(P_k \xi^k_{\;;i}\, u^i - u_{i;k} P^i \xi^k) = 0. \quad ...(3.2)$$

Using $\xi^k = \varphi\omega^k$ and the identities $\omega^k_{\;;k} + 2\omega^k D u_k = 0$,

$$D\omega^k = - \omega^j D u_j u^k + \tfrac{1}{2}\eta^{kijm} u_i D u_{j;m} - 2\theta\omega^k + \sigma^k_m \,\omega^m$$

and

$$D\omega^2 = - 4\theta\omega^2 - 2\sigma_{ij}\omega^i\omega^j + \omega^{ij} D u_{i;j}$$

in (3.2) we observe that the last term of (3.2) vanishes if $P^i = \lambda\omega^i$, where λ is a non-zero arbitrary function and $D\omega^2 = 2\{D \ln \varphi - 11\theta\}$. Consequently (3.2) yields

$$(\mu_0\omega^k)_{;k} = 0 \quad ...(3.3)$$

which is equivalent to

$$(\alpha\mu_0\omega^k)_{;k} = 0; \; \alpha_{,k}\omega^k = 0 \quad ...(3.4)$$

which proves the statement.

To interpret this theorem we use the relation (Greenberg 1970)

$$\underset{V}{D} \ln A = - \underset{V}{D} \ln \mid \omega \mid + \omega^k D u_k / \mid \omega \mid \quad ...(3.5)$$

and the identity $\omega^{k}_{;k} + 2\omega^{k}Du_{k} = 0$ in (3.3), then we get

$$\underset{\vee}{D}(\mu_0^{1/2} A \mid \omega \mid) = 0 \qquad ...(3.6)$$

where A is the proper area subtended by the vortex lines as they pass through the screen in the 2-surface dual to the surface formed by u_i and ω_i. $\underset{\vee}{D}$ stands for the absolute derivative along the vortex lines. Equation (3.6) reveals that $(\rho + 3p + \mu \mid h \mid^2 + \lambda \mid e \mid^2)^{1/2} A \mid \omega \mid$ is constant along the vortex tubes. This result may be regarded as a generalization of the Kelvin-Helmholtz theorem in case of electromagnetic fluids.

Theorem 3.2 — For an electromagnetic fluid with 'frozen-in' magnetic fields admitting the symmetry property $\underset{\xi}{\mathcal{L}} R_{ij} = 0$ for $\xi^i = \varphi \bar{B}^i$, the conservation law $(\mu_0^{1/2} B^k)_{;k} = 0$ holds if the 'energy-flux' vector is parallel or antiparallel to the electric fields.

PROOF : The symmetry condition $u^i u^j \underset{\xi}{\mathcal{L}} R_{ij} = 0$ is equivalent to

$$\mu_{0,k} B^k - 2\mu_0 B^k Du_k + 2\,[P_k DB^k - u_{i;k} P^i B^k] = 0. \qquad ...(3.7)$$

Using (2.8) and (2.9) for $a_i = e_i / \mid e \mid$, $\mid e \mid > 0$, in the resulting equation obtained by the contraction of (2.1) with P_i, we get

$$(P_i DB^i - u_{i;j} P^i B^j) + 3\theta P_i B^i - \eta^{ijkl} P_i u_j Du_k e_l$$
$$+ (\ln \mid e \mid)_{,j}\, \eta^{ijkl} P_i u_k e_l + \eta^{ijkl} P_i u_k e_j \hat{D} a_l = 0 \qquad ...(3.8)$$

which reduces to

$$P_i DB^i - u_{i;j} P^i B^j = 0 \qquad ...(3.9)$$

when $P_i = \lambda e_i$, where λ is a non-zero arbitrary function.

Combining (3.7) and (3.9), we have

$$\mu_{0,k} B^k - 2\mu_0 B_k Du^k = 0. \qquad ...(3.10)$$

Now the electromagnetic fluid with 'frozen-in' magnetic fields satisfies the identity (Prasad 1978d)

$$B^k_{;k} + B_k Du^k = 0. \qquad ...(3.11)$$

In view of (3.10) and (3.11), we obtain

$$(\mu_0^{1/2} B^k)_{;k} = 0 \qquad ...(3.12)$$

which proves the statement.

Theorem 3.3 — For an electromagnetic fluid with 'frozen-in' electric fields admitting symmetry property $\underset{\xi}{\mathcal{L}} R_{ij} = 0$ for $\xi^i = \varphi D^i$. the conservation law $(\mu_0^{1/2} D^k)_{;k} = 0$ holds if the energy-flux vector is parallel or anti-parallel to the magnetic fields.

The proof of the theorem runs on the lines of the proof of Theorem 3.2.

Now we observe a few examples of conformal motions (CM) which degenerate into motions (M) under restrictive case in the space-time of the electromagnetic fluids. The infinitesimal transformation $\bar{x}^i = x^i + \xi^i \delta t$ defines the conformal motion if $\underset{\xi}{\mathcal{L}} g_{ij} = \Lambda g_{ij}$, where Λ is non-zero scalar function, and motion if $\underset{\xi}{\mathcal{L}} g_{ij} = 0$.

Theorem 3.4 — The conformal motion in the space-time of an electromagnetic fluid with respect to the electric field vector degenerates into motion when the electrical conductivity of the fluid is constant.

PROOF : The conformal motion with respect to the electric field vector e^i is given by

$$e^i_{;i} = 2\Lambda = -2e^i Du_i. \quad \ldots(3.13)$$

The conservation of electric current yields

$$(\epsilon u^i)_{;i} + \bar{k}_{,i} e^i + \bar{k} e^i_{;i} = 0 \quad \ldots(3.14)$$

where ϵ is the charge density and $\bar{k}$ the electrical conductivity of the fluid. The conservation of charge density is given by

$$(\epsilon u^i)_{;i} = 0. \quad \ldots(3.15)$$

In view of (3.14) and (3.15) we conclude that

$$e^i_{;i} = 0 \quad \ldots(3.16)$$

when k is constant. Consequently $\Lambda = 0$ and this proves the statement.

Theorem 3.5 — The conformal motion in the space-time of an electromagnetic fluid with respect to the magnetic field vector degenerates into motion when the magnetic field tubes are parallel or anti-parallel to the axis of rotation of the electromagnetic fluid which is in 'steady rigid rotation'.

PROOF : The conformal motion with respect to the magnetic field vector is given by

$$h^i_{;i} = 2\Lambda = -2h^i Du_i. \quad \ldots(3.17)$$

Using the fact (Prasad 1978d) [i.e. the magnetic field and fluid acceleration are orthogonal when the magnetic field tubes are parallel or anti-parallel to the axis of rotation of the electromagnetic fluid which is in 'steady rigid rotation'] in (3.17) we prove the statement.

REFERENCES

Asgekar, G. G., and Date, T. H. (1977). Collineations and motions in self-gravitating magnetofluids. *J. Math. Phys.*, **18**, 738.

Davis, W. R. (1974). Conservation laws in Einstein's general theory of relativity in Lanczos Festschrift, edited by B.K.P. Scaife (London and N.Y.). p. 29; Symmetry properties and conservation laws in relativistic continuum mechanics, in Proc. of symposium of symmetry, similarity, and group theoretic methods in mechanics, edited by P. G. Glockner and M. C. Singh (Calgary, Alta), p. 119.

Davis, W. R., Green, L. H., and Norris, L. K. (1976). Relativistic matter fields admitting Ricci collineations and related conservation laws. *Nuovo Cim.*, **34B**, 256.

Ehlers, J. (1961). *Akad. Wiss. Lit. Mainz. Abh. Math. Nat. K*1, **11**.

Glass, E. N. (1975). Weyl tensor and shear-free perfect fluids. *J. Math. Phys.*, **16**, 2361.

Greenberg, P. J. (1970). The general theory of space-like congruence with an application to vorticity. *J. Math. Analysis Applic.*, **30**, 128.

Oliver, D. R., and Davis, W. R. (1976). Perfect fluids and symmetry mappings leading to conservation laws. *J. Math. Phys.*, **17**, 1790.

Prasad, G. (1978a). Relativistic magnetofluids and symmetry mappings I. *Indian J. pure appl. Math.*, **9**, 682.

——— (1978b). Relativistic magnetofluids and symmetry mappings II. *Indian J. pure appl. Math.*, **9**, 692.

——— (1978c). On the geometry of relativistic electromagnetic fluid flows with geometrical symmetries. *Indian J. pure appl. Math.*, **9**, 457.

——— (1978d). On the geometry of relativistic electromagnetic fluid flows. (Communicated).

Shaha, R. R. (1974). Definite magnetofluid scheme in general relativity. *Ann. Inst. Henri Poincare*, **20**, 189.

Prasad, G., and Sinha, B. B. (1978). Relativistic magnetofluids and symmetry mappings III. *Indian J. pure appl. Math.*, **9**, 893.