*Indian J. pure appl. Math.*, **10**(8) : 1002-1008, August 1979

# ASYMMETRIC VIBRATION OF CIRCULAR SANDWICH PLATES

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(*Received* 13 *November* 1978)

Free asymmetric vibration of circular sandwich plates considered in the present paper takes into account the transverse shear, rotatory inertia and flexural rigidity of the core as well as of the facings. The solution of the equations of motion is obtained in terms of Bessel functions with the aid of auxiliary functions. The frequencies computed for clamped edge conditions are compared with those of homogeneous plates.

## INTRODUCTION

A considerable amount of work has been done on vibration of beams and plates of sandwich constructions but a very few papers are available on circular sandwich plates. Yu and Koplik (1967) have considered the torsional vibrations of circular sandwich plates by taking facings as membranes. Pujara (1969) has considered the vibrations of circular sandwich plates by neglecting the shear strain in the facings and the core having no membrane or bending resistance. Axisymmetric vibration of circular sandwich plates is considered by Mirza and Singh (1974).

In the present paper, free asymmetric vibration of circular sandwich plates is studied by adopting the theory given by Yu (1959). The effects of transverse shear deformation and rotatory intertia are taken into account for the core and the facings by taking the angle of shear for facings different from that of the core. Thus the theory is applicable even to cases where the facings have appreciable thickness compared to the core, and holds for any ratio of material densities and elastic constants of the core and the facings. The equations of motion derived by Hamilton's principle are solved in terms of Bessel functions with the aid of auxiliary variables. The frequency parameter is computed for a clamped plate for first four normal modes of vibration and for first three circumferencial wave numbers and compared with those of homogeneous plates.

## DISPLACEMENTS AND STRAINS

We consider a circular sandwich plate of radius $a$ and thickness $2h$. The thickness of the core and each of the two facings are taken $2h_1$ and $h_2$, so that $h = h_1 + h_2$. The plate is referred to cylindrical coordinates $r, \theta, z$, by taking the axis of the plate as the line $r = 0$ and the middle plane, the lower and upper interfaces, and the bottom and top in the planes $z = 0, \mp h_1, \mp h$ respectively. The two facings have same material but different from that of core. The various quantities taken for the core, the lower facing and the upper facing will be distinguished by the subscripts 1, 2 and 3 respectively.

The displacement components $u_i$, $v_i$ and $w_i$, $i = 1, 2, 3$, in the directions $r$, $\theta$ and $z$ respectively, are taken to be

$$\left.\begin{aligned} & u_1 = z\beta_1,\ v_1 = z\gamma_1,\ w_i = w \\ & u_2, u_3 = \mp h_1\beta_1 + (z \pm h_1)\,\beta_2;\ v_2, v_3 = \mp h_1\gamma_1 + (z \pm h_1)\,\gamma_2 \end{aligned}\right\} \quad \ldots(1)$$

where $u_i$, $v_i$ and $w_i$ are the functions of $r, \theta, z$ and $t$ but $\beta_1$, $\beta_2$, $\gamma_1$, $\gamma_2$ and $w$ are the functions of $r$, $\theta$ and $t$ only. The functions $\beta_1$, $\beta_2$ and $\gamma_1$, $\gamma_2$ are the rotations in the $rz$- and $\theta z$-planes of the normals to the transverse sections of the core and the facings and $t$ denotes the time.

The strain components, obtained by using the expressions given by Love (1944), are

$$\left.\begin{aligned} & \epsilon_{r1} = z\beta_{1,r},\ \epsilon_{\theta 1} = z(\beta_1 + \gamma_{1,\theta})/r,\ \epsilon_{zi} = 0 \\ & \epsilon_{r2}, \epsilon_{r3} = \mp h_1(\beta_{1,r} - \beta_{2,r}) + z\beta_{2,r} \\ & \epsilon_{\theta 2}, \epsilon_{\theta 3} = [\mp h_1(\beta_1 - \beta_2 + \gamma_{1,\theta} - \gamma_{2,\theta}) + z(\beta_2 + \gamma_{2,\theta})]/r \\ & \epsilon_{r\theta 1} = z(\beta_{1,\theta} + r\gamma_{1,r} - \gamma_1)/r \\ & \epsilon_{r\theta 2}, \epsilon_{r\theta 3} = [\mp h_1\{\beta_{1,\theta} - \beta_{2,\theta} + r(\gamma_{1,r} - \gamma_{2,r}) - (\gamma_1 - \gamma_2)\} \\ & \qquad\qquad + z(\beta_{2,\theta} + r\gamma_{2,r} - \gamma_2)]/r \\ & \epsilon_{rz1} = \beta_1 + w_{,r},\ \epsilon_{rz2} = \epsilon_{rz3} = \beta_2 + w_{,r} \\ & \epsilon_{\theta z1} = \gamma_1 + w_{,\theta}/r,\ \epsilon_{\theta z2} = \epsilon_{\theta z3} = \gamma_2 + w_{,\theta}/r \end{aligned}\right\} \quad \ldots(2)$$

where a comma followed by a suffix denotes the differentiation with respect to that variable.

## EQUATIONS OF MOTION

By applying Hamilton's energy principle, following Gupta (1970), the equations of motion are obtained to be

$$P_{1,r} + \frac{1}{r}(P_1 + P_{3,\theta} - P_4) = \tfrac{2}{3}h_1^2(\rho_1 h_1 + 3\rho_2 h_2)\,\beta_{1,tt} + \rho_2 h_1 h_2^2\,\beta_{2,tt} \quad \ldots(3)$$

$$P_{3,r} + \frac{1}{r}(P_3 + P_{2,\theta} - P_5) = \tfrac{2}{3}h_2^2(\rho_1 h_1 + 3\rho_2 h_2)\,\gamma_{1,tt} + \rho_2 h_1 h_2^2\,\gamma_{2,tt} \quad \ldots(4)$$

$$P_{6,r} + \frac{1}{r}(P_6 + P_{8,\theta} - P_9) = \rho_2 h_2^2(h_1\beta_{1,tt} + \tfrac{2}{3}h_2\beta_{2,tt}) \quad \ldots(5)$$

$$P_{8,r} + \frac{1}{r}(P_8 + P_{7,\theta} - P_{10}) = \rho_2 h_2^2 (h_1\gamma_{1,tt} + \tfrac{2}{3} h_2\gamma_{2,tt}) \qquad \ldots(6)$$

$$Q_{r,r} + \frac{1}{r}(Q_r + Q_{\theta,\theta}) = 2(\rho_1 h_1 + \rho_2 h_2)\, w_{,tt} \qquad \ldots(7)$$

where $\rho_1$ and $\rho_2$ are densities of the core and the facings and

$$\left.\begin{aligned} &P_1 = M_{r1} - N_1,\ P_2 = M_{\theta 1} - N_2,\ P_3 = M_{r\theta 1} - N_3,\\ &P_4 = P_2 + rQ_{r1},\ P_5 = rQ_{\theta 1} - P_3,\\ &P_6 = M_1 + N_1,\ P_7 = M_2 + N_2, P_8 = M_3 + N_3,\\ &P_9 = P_7 + rQ_1,\ P_{10} = rQ_2 - P_8,\ Q_r = Q_{r1} + Q_{r2} + Q_{r3},\\ &Q_\theta = Q_{\theta 1} + Q_{\theta 2} + Q_{\theta 3},\ N_1 = h_1(N_{r2} - N_{r3}),\\ &N_2 = h_1(N_{\theta 2} - N_{\theta 3}),\ N_3 = h_1(N_{r\theta 2} - N_{r\theta 3}),\\ &M_1 = M_{r2} + M_{r3},\ M_2 = M_{\theta 2} + M_{\theta 3},\ M_3 = M_{r\theta 2} + M_{r\theta 3},\\ &Q_1 = Q_{r2} + Q_{r3},\ Q_2 = Q_{\theta 2} + Q_{\theta 3}. \end{aligned}\right\} \qquad \ldots(8)$$

The stress resultants are given by

$$(N_{ri}, N_{\theta i}, N_{r\theta i}, Q_{ri}, Q_{\theta i}, M_{ri}, M_{\theta i}, M_{r\theta i})$$
$$= \int_{x_i}^{y_i} (\sigma_{ri}, \sigma_{\theta i}, \sigma_{r\theta i}, k_s\sigma_{rzi}, k_s\sigma_{\theta zi}, z\sigma_{ri}, z\sigma_{\theta i}, z\sigma_{r\theta i})\, dz \qquad \ldots(9)$$

where the limits of integration $x_i$ to $y_i$, $i = 1, 2, 3$, stand for $-h_1$ to $h_1$, $-h$ to $-h_1$ and $h_1$ to $h$ respectively, and $k_s$ is the Timoshenko constant.

The stress-strain relations used for obtaining the above equations of motion in terms of displacements are

$$\left.\begin{aligned} &\sigma_{ri} = \lambda_i\epsilon_{ri} + \lambda_i'\epsilon_{\theta i},\ \sigma_{\theta i} = \lambda_i\epsilon_{\theta i} + \lambda_i'\epsilon_{ri}\\ &(\sigma_{r\theta i}, \sigma_{rzi}, \sigma_{\theta zi}) = \mu_i(\epsilon_{r\theta i}, \epsilon_{\theta zi}, \epsilon_{rzi})\\ &\lambda_i = E_i/(1 - \nu_i^2),\ \lambda_i' = \nu_i\lambda_i,\ \mu_i = E_i/2(1 + \nu_i) \end{aligned}\right\} \qquad \ldots(10)$$

where $E_i$ and $\nu_i$ are Young's modulii and Poisson's ratios.

## Introduction of Auxiliary Functions

For getting a simpler system of equations we introduce auxiliary functions in the following manner :

$$\left.\begin{aligned} &\beta_i = (G_{i,r} + mH_i/r)\cos m\theta \exp(i\omega t)\\ &\gamma_i = -(H_{i,r} + mG_i/r)\sin m\theta \exp(i\omega t)\\ &w = W\cos m\theta \exp(i\omega t) \end{aligned}\right\} \qquad \ldots(11)$$

where $G_i, H_i$, $i = 1, 2$, are functions of $r$ alone, $\omega$ is angular frequency and $m$ is circumferencial wave number.

We obtain two equations, first by operating eqn. (3) by $(\partial/\partial r) + (1/r)$ and eqn. (4) by $(1/r)(\partial/\partial\theta)$ and then adding, and second by operating eqn. (4) by $(\partial/\partial r) + (1/r)$ and eqn. (3) by $(1/r)(\partial/\partial\theta)$ and then subtracting. Expressing these two equations in terms of displacements, then using relations (11) and making them non-dimensional, we get

$$(a_1 D^2 + b_2) D^2 G_1 + (a_3 D^2 + b_4) D^2 G_2 + b_1 D^2 W = 0 \qquad ...(12)$$

$$(a_7 D^2 + b_2) D^2 H_1 + (a_8 D^2 + b_4) D^2 H_2 = 0. \qquad ...(13)$$

Applying the same method, eqns. (5) and (6) can be reduced to the form:

$$(a_2 D^2 + b_3) D^2 G_1 + (a_4 D^2 + b_6) D^2 G_2 + b_5 D^2 W = 0 \qquad ...(14)$$

$$(b_8 D^2 + b_3) D^2 H_1 + (a_9 D^2 + b_6) D^2 H_2 = 0. \qquad ...(15)$$

Lastly, expressing eqn. (7) in terms of displacements, then using relations (11) and then making it non-dimensional, we get

$$k_s D^2 G_1 + a_6 D^2 G_2 + (a_5 D^2 + b_7) W = 0. \qquad ...(16)$$

Eliminating $G_1$ and $G_2$ from eqns. (12), (14) and (16), we get

$$(r_0 D^6 + r_1 D^4 + r_2 D^2 + r_3) W = 0. \qquad ...(17)$$

Same equation will be obtained for $G_1$ or $G_2$ by eliminating the other two. Now, eliminating $H_2$ from eqns. (13) and (15), we get

$$(r_4 D^4 + r_5 D^2 + r_6) H_1 = 0. \qquad ...(18)$$

Same equation will be obtained for $H_2$ if we eliminate $H_1$.

In the above eqns. (12) to (18), various symbols used are

$$\left.\begin{aligned}
&a_1 = 2a_2 + 2R_1/3,\ a_2 = R_h R_2,\ a_3 = R_h a_2,\ a_4 = 2a_3/3,\\
&a_5 = k_s + a_6,\ a_6 = k_s b_8,\\
&a_7 = 2b_8 + 2/3,\ a_8 = R_h b_8,\ a_9 = 2a_8/3,\ b_1 = -\,2k_s/R_a^2,\\
&b_2 = b_1 + 2b_3 + 2b_9/3,\ b_3 = R_h R_\rho b_9,\\
&b_4 = R_h b_3,\ b_5 = R_\mu b_1,\ b_6 = b_5 + 2b_4/3,\ b_7 = b_3 + b_9,\\
&b_8 = R_h R_\mu,\ b_9 = R_1 \Omega^2,
\end{aligned}\right|$$

*(equations continued on p. 1006)*

$$\left.\begin{aligned}
r_0 &= a_5c_1,\ r_1 = a_5c_2 + b_7c_1 + a_6(b_1a_2 - a_1b_5) \\
&\qquad\qquad + k_s(a_3b_5 - b_1a_4), \\
r_2 &= a_5c_3 + b_7c_2 + a_6(b_1b_3 - b_2b_5) + k_s(b_4b_5 - b_1b_6), \\
r_3 &= b_7c_3, \\
c_1 &= a_1a_4 - a_2a_3,\ c_2 = a_1b_6 - b_2a_4 - a_2b_4 - a_3b_3, \\
c_3 &= b_2b_6 - b_3b_4, \\
r_4 &= a_7a_9 - b_8a_8,\ r_5 = a_7b_6 + a_9b_2 - b_8b_4 - a_8b_3, \\
r_6 &= b_2b_6 - b_3b_4, \\
R_a &= h_1/a,\ R_h = h_2/h_1,\ R_\mu = \mu_2/\mu_1,\ R_\rho = \rho_2/\rho_1,\ R_1 = \lambda_1/\mu_1, \\
R_2 &= \lambda_2/\mu_1,\ R = r/a, \\
D^2 &\equiv (\partial^2/\partial R^2) + (\partial/R\partial R) - (m^2/R^2) \quad \text{and} \quad \Omega^2 = \rho_1 a^2\omega^2/\lambda_1
\end{aligned}\right\} \quad \ldots(19)$$

where $\Omega$ is the non-dimensional frequency parameter.

## Solution

It is clear from eqns. (17) and (18) that the solutions for auxiliary functions can be taken as

$$(W, G_1, G_2) = \sum_{\alpha=1}^{3} (1, g_\alpha, k_\alpha) A_\alpha J_m(Rn_\alpha) \quad \ldots(20)$$

$$(H_1, H_2) = \sum_{\alpha=4}^{5} (1, s_\alpha) A_\alpha J_m(Rn_\alpha) \quad \ldots(21)$$

where $J_m$ is the Bessel function of order $m$ of first kind, $A_\alpha$ are the arbitrary constants, $n_\alpha^2$ ($\alpha = 1, 2, 3$) and $n_\alpha^2$ ($\alpha = 4, 5$) are the roots of the cubic and quadratic equations

$$r_0n^6 - r_1n^4 + r_2n^2 - r_3 = 0 \quad \text{and} \quad r_4n^4 - r_5n^2 + r_6 = 0 \quad \ldots(22)$$

$g_\alpha$ and $k_\alpha$ can be obtained from any two of the eqns. (12), (14) and (16) after substituting solution (20), and $s_\alpha$ can be obtained from eqn. (13) or (15) after substituting solution (21).

## Frequency Equation

If we take the plate clamped at the edge, then the edge conditions are

at $\quad R = 1,\ W = \beta_1 = \beta_2 = \gamma_1 = \gamma_2 = 0. \quad \ldots(23)$

Substituting solutions (20) and (21) in relations (11) and then using the above edge conditions, we get

$$\left.\begin{aligned}
&\sum_{\alpha=1}^{3} A_\alpha J_m(n\alpha) = 0\\
&\sum_{\alpha=1}^{3} A_\alpha g_\alpha J'_m(n_\alpha) + m\sum_{\alpha=4}^{5} A_\alpha J_m(n_\alpha) = 0\\
&\sum_{\alpha=1}^{3} A_\alpha k_\alpha J'_m(n_\alpha) + m\sum_{\alpha=4}^{5} S_\alpha A_\alpha J_m(n_\alpha) = 0\\
&m\sum_{\alpha=1}^{3} A_\alpha g_\alpha J_m(n_\alpha) + \sum_{\alpha=4}^{5} A_\alpha J'_m(n_\alpha) = 0\\
&m\sum_{\alpha=1}^{3} A_\alpha k_\alpha J_m(n_\alpha) + \sum_{\alpha=4}^{5} A_\alpha S_\alpha J'_m(n_\alpha) = 0
\end{aligned}\right\} \qquad \ldots(24)$$

where

$$J'_m(n_\alpha) = \left[\frac{d}{dR}\{J_m(Rn_\alpha)\}\right]_{R=1} = n_\alpha J_{m-1}(n_\alpha) - mJ_m(n_\alpha). \qquad \ldots(25)$$

Eliminating the arbitrary constants from the above equations, we get the frequency equation, say

$$|\, D_{ij} \,| = 0,\ (i, j = 1 \text{ to } 5) \qquad \ldots(26)$$

where the elements $D_{ij}$ of the above determinant involve the frequency parameter $\Omega$ implicitly. The frequency equation for a homogeneous plate can be obtained by putting $R_h = 0$ in the equations of motion and then proceeding in a similar manner.

## Numerical Results and Discussions

In the present investigation, $k_s$ is taken equal to $\pi^2/12$ and the materials for the core and the facings are taken to be cellulose acetate and aluminium respectively, for which the values of various constants are:

$$R_\rho = 34.4,\ R_\mu = 1683.0,\ R_1 = 2.20,\ R_2 = 4790.0,\ \nu_1 = 0.0909091,$$
$$\nu_2 = 0.297286.$$

The relation between $\Omega$ and $R_h$ is shown in Fig. 1 for first four modes of vibration for $m = 1, 2, 3$. It is seen that the value of $\Omega$ increases with the increase in $R_h$, first slowly and then rapidly, in all the modes of vibration and for all values of $m$. Also, the values as well as the rate of change of $\Omega$ increase as we go to higher modes of vibration or higher values of $m$. In Fig. 2 is plotted $\Omega$ versus $R_a$ for sandwich and homogeneous plates for first four modes for $m = 1$ and for first two modes for $m = 2, 3$. It is clear that $\Omega$ increases with the increase in $R_a$ for both type of plates but the rate of increase for homogeneous plates is greater than that for sandwich plates. The difference in the values of $\Omega$ for any two successive modes is also greater for homogeneous plates than that for sandwich plates.

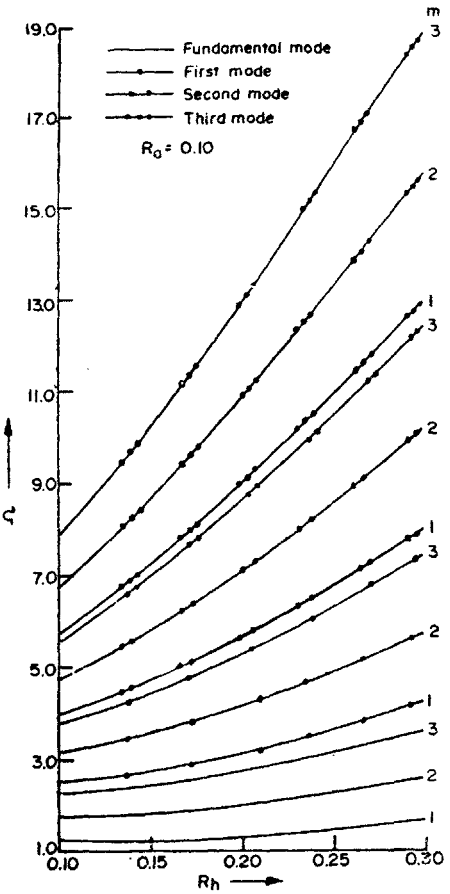


FIG. 1. Ω vs $R_h$ for sandwich plate.

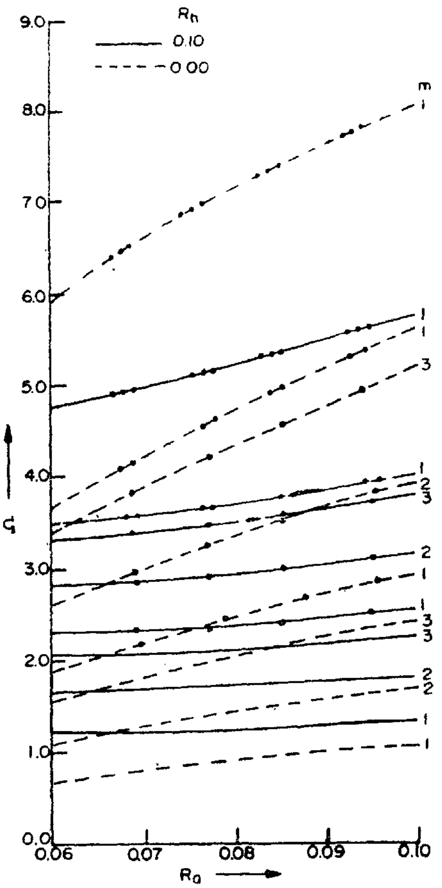


FIG. 2. Ω vs $R_a$ for sandwich and homogeneous plates.

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