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INFINITESIMAL VARIATIONS OF HYPERSURFACES OF AN ALMOST PRODUCT AND ALMOST DECOMPOSABLE MANIFOLD

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In this paper we have studied the infinitesimal variations of the structure tensors of the almost paracontact structure induced on the hypersurface of the almost product and almost decomposable manifold under various conditions. Infinitesimal variations of the induced connexion and second fundamental form have also been studied in order to deduce a few interesting results.

1. INTRODUCTION

A differentiable manifold M_n equipped with a tensor field F of type (1, 1) and a Riemannian metric G satisfying

$$F^2 = I,\ G(F\tilde{X}, F\tilde{Y}) = G(\tilde{X}, \tilde{Y}), \qquad \ldots(1.1)$$

$\tilde{X}, \tilde{Y}$ being arbitrary vector fields on M_n, is called an almost product Riemannian manifold. Further, if the Riemannian connexion E induced by G renders the structure tensor F parallel, M_n is called an almost product and almost decomposable manifold.

Let us embed a hypersurface M_{n-1} into M_n by the isometric immersion $b : M_{n-1} \rightarrow M_n$. Correspondingly we have the Jacobian b_* of b, denoted by B which carries $T_p(M_{n-1})$ into $T_{b(p)}(M_n)$ injectively. Since the immersion is isometric we have

$$G(BX, BY) = g(X, Y) \qquad \ldots(1.2)$$

g being the metric induced on the hypersurface and X, Y denoting arbitrary vector fields on M_{n-1}. Denoting the unit normal field to M_{n-1} by N we have

$$G(BX, N) = 0 \qquad \ldots(1.3)$$

$$G(N, N) = 1 \qquad \ldots(1.4)$$

We write the transformation equation as

$$BfX = F(BX) - A(X)\, N \qquad \ldots(1.5)$$

where f is a tensor field of type (1, 1) and A is a 1-form on M_{n-1}. It is possible to obtain a map

$$B^{-1} : T_{b(p)}(M_n) \to T_p(M_{n-1})$$

such that $B^{-1}B = I$, $BB^{-1} = I - N \otimes N^*$, $B^{-1}N = 0$ (Blair and Ludden 1969), where N^* is the 1-form associate to N. From (1.5) we obtain

$$f^2X = X - A(X)\, B^{-1}FN. \qquad \text{...(1.6)}$$

By setting $FN = BT + aN$(a being a scalar function) in the above equation we find

$$f^2X = X - A(X)\, T \qquad \text{...(1.7)}$$

which is the almost paracontact structure (Sato 1976). It is easy to show that

$$fT = 0,\ A(fX) = 0,\ A(T) = 1. \qquad \text{...(1.8)}$$

It can be proved that $a = 0$ and thus

$$FN = BT. \qquad \text{...(1.9)}$$

The metric g is found to satisfy

$$g(fX, fY) = g(X, Y) - A(X)\, A(Y). \qquad \text{...(1.10)}$$

Consequently an almost paracontact Riemannian structure (f, T, A, g) gets induced on M_{n-1}.

If D is the Riemannian connexion induced on M_{n-1} by g, we have the Gauss and Weingarten formulae

$$E_{BX}BY = BD_XY + h(X, Y)\, N \qquad \text{...(1.11)}$$

$$E_{BX}N = -BHX \qquad \text{...(1.12)}$$

where h is the second fundamental form of M_{n-1} and H is a tensor field of type (1, 1) associated to h. If K and $\tilde{K}$ stand for the curvature tensors of the hypersurface and the enveloping manifold, we have the Gauss and Codazzi equations

$$'\tilde{K}(BX, BY, BZ, BU) = {}'K(X, Y, Z, U) - h(Y, Z)\, h(X, U) + h(X, Z)\, h(Y, U) \qquad \text{...(1.13)}$$

$$'\tilde{K}(BX, BY, BZ, N) = (D_Xh)\,(Y, Z) - (D_Yh)\,(X, Z) \qquad \text{...(1.14)}$$

where $'K$ and $'\tilde{K}$ are the associate covariant curvature tensors of M_{n-1} and M_n.

Now let us differentiate eqn. (1.5) along the hypersurface and use $E_{\tilde{X}}F = 0$ to get

$$E_{BY}BfX = F(E_{BY}BX) - \{(D_YA)\,(X) + A(D_YX)\}\, N - A(X)\, E_{BY}N.$$

Use of (1.9), (1.11) and (1.12) reduces it to

$$B\{(D_Y f)\, X - h(X, Y)\, T - A(X)\, HY)\} + \{(h(fX, Y) + (D_Y A)\, X\}\, N = 0$$

whose tangential and normal components are

$$(D_Y f)\, X = h(X, Y)\, T + A(X)\, HY \qquad \text{...(1.15)}$$

$$(D_Y A)\, X = -h(fX, Y). \qquad \text{...(1.16)}$$

Covariant differentiation of (1.9) along M_{n-1} yields

$$D_X T = -\, fHX \qquad \text{...(1.17)}$$

which could have been straightaway obtained from

$$A(X) = g(T, X) \quad \text{and} \quad g(fX, Y) = g(X, fY).$$

The almost paracontact structure (f, T, A) is said to be normal if

$$[f, f]\,(X, Y) - (dA)\,(X, Y)\, T = 0$$

where d is the operator of exterior derivation and $[f, f]$ is the Nijenhuis torsion of f defined by

$$[f, f]\,(X, Y) = [fX, fY] + f^2[X, Y] - f[fX, Y] - f[X, fY].$$

Thus the normality condition of (f, T, A) takes the form

$$(D_{fX} f)\, Y - (D_{fY} f)\, X + f\,\{(D_Y f)\, X - (D_X f)\, Y\}$$
$$- \{(D_X A)\, Y - (D_Y A)\, X\} = 0.$$

If the almost paracontact structure induced on M_{n-1} be normal we obtain from the above condition using (1.15) and (1.16)

$$A(X)\,\{(Hf - fH)\, Y\} - A(Y)\,\{(Hf - FH)\, X\} = 0$$

which, in virtue of (1.8) and (1.10), yields

$$Hf = fH \qquad \text{...(1.18)}$$

whence it follows that

$$h(T, T)\, T = HT \qquad \text{...(1.19)}$$

showing that $h(T, T)$ is an eigenvalue of H and the corresponding eigen vector is T. Let us denote $h(T, T)$ by τ.

The almost paracontact Riemannian structure is called paracontact Riemannian structure if

$$(D_X A)\, Y + (D_Y A)\, X = 2\, {}'f(X, Y) \qquad \text{...(1.20)}$$

where $\quad {}'f(X, Y) = g(fX, Y).$

The sectional curvature of the normal paracontact Riemannian manifold with respect to a plane section containing T is found, on calculation, to be -1.

More generally, we assume (Sato 1977)

$$(D_X A)(Y) + (D_Y A)(X) = 2\alpha\,'f(X, Y) \quad ...(1.21)$$

in a normal almost paracontact hypersurface of M_n.

Applying (1.16) to the above equation, we have

$$Hf = fH = -\alpha f \quad ...(1.22)$$

whence we obtain

$$HX = -\alpha X + (\tau + \alpha)\, A(X)\, T \quad ...(1.23)$$

Equations (1.15), (1.16) and (1.17) then transform as

$$(D_X f)\, Y = -\alpha(g(X, Y)\, T + A(Y)\, X) + 2(\tau + \alpha)\, A(X)\, A(Y)\, T \quad ...(1.24)$$

$$(D_X A)\, Y = \alpha\,'f(X, Y) \quad ...(1.25)$$

$$D_X T = \alpha f X \quad ...(1.26)$$

From (1.24) and (1.26), for constant α, we have

$$K(X, Y, T) = -\alpha^2\{A(Y)\, X - A(X)\, Y\}$$

which reveals that for a normal almost paracontact hypersurface with (1.21) involving constant α, the sectional curvature with respect to a plane section containing T is $-\alpha^2$.

Let us call such a structure a normal paracontact structure with f-sectional curvature $-\alpha^2$.

2. Infinitesimal Variation of a Hypersurface of an Almost Product and Almost Decomposable Manifold

Suppose that the infinitesimal variation of the hypersurface is brought about by the restriction of an almost decomposable Killing vector field U on the enveloping manifold to the hypersurface. Accordingly the variation of the differential of imbedding is given by (Yano 1977)

$$(\delta B)(X) = \epsilon E_{BX} U \quad ...(2.1)$$

where ϵ is an infinitesimally small number.

Splitting U into its tangential and normal components as

$$U = BV + \lambda N$$

and using (1.11) and (1.12) we express (2.1) as

$$(\delta B)(X) = \epsilon\{B(D_X V - \lambda HX) + (X\lambda + h(X, V)) N\}. \qquad ...(2.2)$$

The infinitesimal variation of N is given by (Yano 1957)

$$\delta N = \epsilon L_U N = \epsilon BW$$

$L_U N$, the Lie derivative of N being orthogonal to N. Infinitesimal variation of eqn. (1.3) yields

$$G(BD_X V + h(X, V) N + (X\lambda) N - \lambda BHX, N) = -G(BX, BW)$$

which implies

$$W = -(HV + \Lambda)$$

where Λ stands for the vector field associate to the gradient of λ. Thus we obtain

$$\delta N = -\epsilon B(HV + \Lambda). \qquad ...(2.3)$$

Now varying eqn. (1.5) infinitesimally we find

$$(\delta B)(fX) + B(\delta f) X = F((\delta B) X) - (\delta A)(X) N - A(X)\, \delta N.$$

Making use of (1.5), (2.2) and (2.3) in it we find

$$\begin{aligned} B(\delta f) X + (\delta A)(X) N = \epsilon\{&B(f(D_X V - \lambda HX)) + A(D_X V - \lambda HX) N \\ &+ (h(X, V) + X\lambda) BT - BD_{fX} V + \lambda BHfX \\ &- h(fX, V) N - (fX)\lambda N + A(X) B(HV + \Lambda)\}. \end{aligned}$$

Equating the tangential and normal components we have

$$\begin{aligned} (\delta f) X = \epsilon\{f(D_X V - \lambda HX) + (h(X, V) + X\lambda) T - D_{fX} V \\ + \lambda HfX + A(X)(HV + \Lambda)\} \end{aligned} \qquad ...(2.4)$$

$$(\delta A)(X) = \epsilon\{A(D_X V - \lambda HX) - h(fX, V) - (fX)\lambda\}. \qquad ...(2.5)$$

Since the Lie derivative of f along V is given by

$$\begin{aligned} (L_V f) X &= L_V(fX) - f(L_V X) \\ &= D_V(fX) - D_{fX} V - f(D_V X - D_X V) \end{aligned}$$

eqn. (2.4) assumes the form

$$\begin{aligned} (\delta f) X = \epsilon\{(L_V f) X + \lambda(Hf - fH) X + (h(X, V) + X\lambda) T \\ + A(X)(HV + \Lambda) - (D_V f) X\}. \end{aligned}$$

Using (1.15) in the above equation we have

$$(\delta f) X = \epsilon\{(L_V f) X + \lambda(Hf - fH) X + (X\lambda) T + A(X)\Lambda\}. \qquad ...(2.6)$$

Applying eqn. (1.16) and the definition

$$(L_V A)\,X = (D_V A)\,X + A(D_X V)$$

in (2.5) we have

$$(\delta A)\,X = \epsilon\{(L_V A)\,X - \lambda A(HX) - (fX)\,\lambda\}. \qquad \ldots(2.7)$$

Next varying eqn. (1.9) infinitesimally we see that

$$-\epsilon F(B(HV + \Lambda) = B\delta T + (\delta B)\,T$$

which yields by virtue of (1.5) and (2.2)

$$B\delta T + \epsilon\{B(D_T V - \lambda HT) + ((T\lambda + h(T, V))\,N + Bf(HV + \Lambda) + A(HV + \Lambda)\,N\} = 0$$

whose tangential part reduces in virtue of (1.17) to

$$\delta T = \epsilon\{L_V T + \lambda HT - f\Lambda\}. \qquad \ldots(2.8)$$

Lastly, varying eqn. (1.2) infinitesimally we get

$$(\delta g)\,(X, Y) = G((\delta B)\,X, BY) + G(BX, (\delta B)\,Y)$$

which reduces by virtue of (2.2) to

$$(\delta g)\,(X, Y) = \epsilon\{(L_V g)\,(X, Y) - 2\lambda h(X, Y)\}. \qquad \ldots(2.9)$$

Thus we established the following :

Theorem 2.1 — When a hypersurface of an almost product and almost decomposable manifold is varied infinitesimally by means of a vector field $U = BV + \lambda N$, the structure tensors of the almost paracontact hypersurface vary according to eqns. (2.6), (2.7), (2.8) and (2.9).

Corollary 2.1 — When a hypersurface of an almost product and almost decomposable manifold is given infinitesimal tangential variation by means of BV, the variations of the induced almost paracontact structure tensors on the hypersurface are given by their Lie derivatives along V.

Corollary 2.2 — When a hypersurface of an almost product and almost decomposable manifold is given infinitesimal normal variation by means of λN, the variations of the induced almost paracontact structure tensors on the hypersurface are gived by

$$\left.\begin{array}{l} \text{(a)}\quad (\delta f)\,(X) = \epsilon\{\lambda(Hf - fH)\,X + (X\lambda)\,T + A(X)\,\Lambda\} \\ \text{(b)}\quad (\delta A)\,(X) = \epsilon\{\lambda A(HX) + (fX)\,\lambda\} \\ \text{(c)}\quad \delta T = \epsilon\{\lambda HT - f\Lambda\},\ \text{(d)}\quad (\delta g)\,(X, Y) = -2\epsilon\lambda h(X, Y). \end{array}\right\} \qquad \ldots(2.10)$$

Next we say that the infinitesimal variation is parallel when BX and $\bar{B}X$ are both parallel, equivalently when $(\delta B)\, X$ is tangential to the original hypersurface. Since

$$(\delta B)\, X = \epsilon\{B(D_X V - \lambda HX) + (X\lambda + h(X, V)\, N\}$$

therefore for an infinitesimal parallel variation it is necessary and sufficient that

$$X\lambda + h(X, V) = 0. \qquad \text{...(2.11)}$$

Theorem 2.2 — An infinitesimal normal variation of a hypersurface will be parallel iff λ is constant.

Proof is obvious.

Corollary 2.3 — When a hypersurface of an almost product and almost decomposable manifold is given infinitesimal normal parallel variation, the structure tensors f, T, A and g of the hypersurface vary as

$$\left.\begin{array}{l} \text{(a)}\ \ (\delta f)\,(X) = \epsilon\lambda(Hf - fH)\, X, \quad \text{(b)}\ \ (\delta A)\,(X) = -\epsilon\lambda A(HX) \\ \text{(c)}\ \ \delta T = \epsilon\lambda HT, \quad \text{(d)}\ \ (\delta g)\,(X, Y) = -2\epsilon\lambda h(X, Y). \end{array}\right\} \qquad \text{...(2.12)}$$

Corollary 2.4 — Let the structure induced on a hypersurface of an almost product and almost decomposable manifold be a normal paracontact structure with f-sectional curvature $-\alpha^2$. Then the infinitesimal normal parallel variation of the hypersurface makes the structure tensors vary as

$$\left.\begin{array}{l} \text{(a)}\ \ (\delta f)\, X = 0, \quad \text{(b)}\ \ (\delta A)\, X = -\epsilon\lambda\tau T, \\ \text{(c)}\ \ \delta T = \epsilon\lambda\tau T, \\ \text{(d)}\ \ (\delta g)\,(X, Y) = -2\epsilon\lambda\{-\alpha g(X, Y) + (\tau + \alpha)\, A(X)\, A(Y)\}. \end{array}\right\} \qquad \text{...(2.13)}$$

3. Variation of the Induced Connexion and Second Fundamental Form of the Hypersurface

Varying infinitesimally the Gauss formula

$$E_{BX} BY = BD_X Y + h(X, Y)\, N$$

we have

$$\epsilon(L_U E)\,(BX, BY) = B\,\{(\delta D)\,(X, Y)\} + (\delta h)\,(X, Y)\, N + h(X, Y)\, \delta N$$

where we have followed the settings $E_{\tilde{X}} \tilde{Y} = E(\tilde{X}, \tilde{Y})$ and $D_X Y = D(X, Y)$. The above equation can be put in the form (Yano 1970)

$$\epsilon\{\tilde{K}(U, BX, BY) + E_{BX} E_{BY} U - E_{E_{BX}BY} U\}$$
$$= B\{(\delta D)\,(X, Y) - \epsilon h(X, Y)\,(HV + \Lambda) + (\delta h)\,(X, Y)\, N$$

where we used (2.3).

Substituting $BV + \lambda N$ for U and using Gauss formula we transform the above equation as

$$B\{(\delta D)(X, Y) - \epsilon h(X, Y)(HV + \Lambda)\} + (\delta h)(X, Y) N$$
$$= \epsilon [B\{D_X D_Y V - D_{D_X Y} V - (D_X \lambda H) Y - ((Y\lambda + h(Y, V)) HX\}$$
$$+ \tilde{K}(BV, BX, BY) + \lambda \tilde{K}(N, BX, BY) + h(X, D_Y V - \lambda HY)$$
$$+ XY\lambda - (D_X Y)\lambda + (D_X h)(Y, V) + h(Y, D_X V)\} N]$$

which is separated into tangential and normal parts as

$$(\delta D)(X, Y) = \epsilon\{D_X D_Y V + K(V, X, Y) - D_{D_X Y} V - (D_Y \lambda H) X$$
$$- (D_X \lambda H) Y + h(X, Y)\Lambda + \lambda h^*(X, Y)\}$$
$$= \epsilon\{(L_V D)(X, Y) - (D_Y \lambda H) X - (D_X \lambda H) Y + h(X, Y)\Lambda$$
$$+ \lambda h^*(X, Y)\}$$

where we set

$$g(h^*(X, Y), Z) = (D_Z h)(X, Y)$$

and

$$(\delta h)(X, Y) = \epsilon\{(D_V h)(X, Y) + \lambda' \tilde{K}(N, BX, BY, N) + h(X, D_Y V)$$
$$- \lambda h(X, HY) + XY\lambda - (D_X Y)\lambda + h(Y, D_X V)\}$$
$$= \epsilon\{(L_V h)(X, Y) - \lambda h(X, HY) + XY\lambda - (D_X Y)\lambda$$
$$+ \lambda' \tilde{K}(N, BX, BY, N)\}.$$

Hence we have the following :

Theorem 3.1 — The infinitesimal variation of the hypersurface brings about the following variations in the connexion and the second fundamental form :

$$(\delta D)(X, Y) = \epsilon\{(L_V D)(X, Y) - (D_Y \lambda H) X - (D_X \lambda H) Y$$
$$+ h(X, Y)\Lambda + \lambda h^*(X, Y)\} \quad \ldots(3.1)$$

and

$$(\delta h)(X, Y) = \epsilon\{(L_V h)(X, Y) - \lambda h(X, HY) + XY\lambda - (D_X Y)\lambda$$
$$+ \lambda' \tilde{K}(N, BX, BY, N)\} \quad \ldots(3.2)$$

Corollary 3.1 — If the infinitesimal variation of the hypersurface were tangential, the variations of D and h would be given by their Lie derivatives along V.

Corollary 3.2 — If the infinitesimal variation of the hypersurface were normal, the variations of D and h would be given by

$$(\delta D)(X, Y) = \epsilon\{h(X, Y) \wedge + \lambda h^*(X, Y) - (D_Y\lambda H) X - (D_X\lambda H) Y\} \quad ...(3.3)$$

$$(\delta h)(X, Y) = \epsilon\{XY\lambda - (D_X Y)\lambda + {}'\tilde{K}(N, BX, BY, N) - \lambda h(X, HY)\} \quad ...(3.4)$$

4. Variation of Normal Paracontact Hypersurface with f-Sectional Curvature $-\alpha^2$

Let us assume that the hypersurface bears a normal paracontact structure with f-sectional curvature $-\alpha^2$.

Theorem 4.1 — A normal paracontact hypersurface with f-sectional curvature $-\alpha^2$ will be varied infinitesimally to a normal paracontact hypersurface with f-sectional curvature $-\alpha^2 - \delta\alpha^2$ iff λ satisfies the differential equation

$$\begin{aligned}\epsilon[XY\lambda &- (D_X Y)\lambda + \lambda\{{}'\tilde{K}(N, BX, BY, N) + \alpha^2(g(X, Y) \\ &- A(X)A(Y)\} + \{(D_T T)\lambda - TT\lambda\} A(X)A(Y) \\ &+ (\tau + \alpha)\{(fX)\lambda A(Y) + (fY)\lambda A(X)\}] \\ &= \{A(X)A(Y) - g(X, Y)\}\delta\alpha. \quad ...(4.1)\end{aligned}$$

Proof: The normal paracontact hypersurface with f-sectional curvature $-\alpha^2$ will be varied infinitesimally to a normal paracontact hypersurface with f-sectional curvature $-\alpha^2 - \delta\alpha^2$ iff

$$\begin{aligned}(\delta h)(X, Y) = &- (\delta\alpha) g(X, Y) - \alpha(\delta g)(X, Y) \\ &+ \{(\delta h)(T, T) + 2h(T, \delta T) + \delta\alpha\} A(X)A(Y) \\ &+ (\tau + \alpha)\{(\delta A)(X)A(Y) + (\delta A)(Y)A(X)\} \quad ...(4.2)\end{aligned}$$

which with the help of eqns. (2.7), (2.8), (2.9), (3.2) and

$$\begin{aligned}(L_V h)(X, Y) = &-\alpha(L_V g)(X, Y) + \{(L_V h)(T, T) + 2h(L_V T, T)A(X)A(Y) \\ &+ (\tau + \alpha)\{(L_V A)(X)A(Y) + A(X)(L_V A)(Y)\}\end{aligned}$$

becomes

$$\begin{aligned}\epsilon\{XY\lambda &- (D_X Y)\lambda + \lambda'\tilde{K}(N, BX, BY, N) - \lambda h(X, HY)\} \\ &= 2\alpha\epsilon\lambda h(X, Y) + \epsilon\{TT\lambda - (D_T T)\lambda - \lambda h(T, HT) \\ &+ 2h(T, \lambda HT - f\Lambda) + (1/\epsilon)\delta\alpha\} A(X)A(Y) - \epsilon(\tau + \alpha)\{(\lambda A(HX) \\ &+ (fX)\lambda) A(Y) + (\lambda A(HY) + (fY)\lambda) A(X)\}.\end{aligned}$$

Now we have

$$h(X, HY) = \alpha^2 g(X, Y) + (\tau^2 + \alpha^2) A(X)A(Y)$$

and in particular

$$h(T, HT) = \tau^2.$$

Consequently the above condition reduces to (4.1).

Conversely if λ satisfies the differential eqn. (4.1), then by retreating the steps we get (4.2).

Corollary 4.1 — The infinitesimal normal parallel variation carries a normal paracontact hypersurface with f-sectional curvature $-\alpha^2$ to a normal paracontact hypersurface with f-sectional curvature $-\alpha^2 - \delta\alpha^2$ iff

$$\lambda\epsilon\{'\tilde{K}(N, BX, BY, N) + \alpha^2(g(X, Y) - A(X)A(Y)\} = \{A(X)A(Y) - g(X, Y)\}\,\delta\alpha. \qquad ...(4.3)$$

Corollary 4.2 — If the enveloping manifold of Corollary (4.1) be flat the condition reduces to

$$\delta\alpha = -\lambda\epsilon\alpha^2. \qquad ...(4.4)$$

Corollary 4.3 — It is impossible to carry a normal paracontact hypersurface (with f-sectional curvature -1) of a flat almost product and almost decomposable manifold over to a normal paracontact hypersurface by an infinitesimal normal parallel variation.

Now let the enveloping manifold be an almost product and almost decomposable manifold of almost constant curvature so that its curvature tensor is given by (Yano 1965)

$$\begin{aligned} '\tilde{K}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) = a\,[&G(\tilde{X}, \tilde{W})\,G(\tilde{Y}, \tilde{Z}) - G(\tilde{X}, \tilde{Z})\,G(\tilde{Y}, \tilde{W}) \\ &+ 'F(\tilde{X}, \tilde{W})\,'F(\tilde{Y}, \tilde{Z}) - F(\tilde{X}, \tilde{Z})\,'F(\tilde{Y}, \tilde{W})] \\ &+ b\,['F(\tilde{X}, \tilde{W})\,G(\tilde{Y}, \tilde{Z}) - 'F(\tilde{X}, \tilde{Z})\,G(\tilde{Y}, \tilde{W}) \\ &+ G(\tilde{X}, \tilde{W})\,'F(\tilde{Y}, \tilde{Z}) - G(\tilde{X}, \tilde{Z})\,'F(\tilde{Y}, \tilde{W})]. \qquad ...(4.5) \end{aligned}$$

For this case the condition (4.3) reduces to

$$\left(a + b + \alpha^2 + \frac{\delta\alpha}{\epsilon\lambda}\right)\{g(X, Y) - A(X)A(Y)\} = b\{g(X, Y) - 'F(X, Y)\}$$

which yields on contraction

$$\left\{a + b + \alpha^2 + \frac{\delta\alpha}{\epsilon\lambda}\right\}(n - 2) = b(n - 1 - f). \qquad ...(4.6)$$

Now contracting eqn. (4.5), we get

$$\widetilde{\text{Ric}}\,(\tilde{Y}, \tilde{Z}) = \{a(n-2) + bf\}\,G(\tilde{Y}, \tilde{Z}) + \{af + b(n-2)\}\,'F(\tilde{Y}, \tilde{Z}) \qquad ...(4.7)$$

which shows that the enveloping manifold is also an almost Einstein manifold (Yano 1965). Let us frame an orthonormal base $\{Be_i, BT, N\}_{i=1,\ldots,n-2}$ of $T_{b(p)}(M_n)$ for any point $p \in M_{n-1}$. Obviously $\{e_i, T\}_{i=1,\ldots,n-2}$ forms an orthonormal base of $T_p(M_{n-1})$. Putting $X = Y = e_i$ and summing over i in eqn. (4.3) we find

$$\epsilon\lambda\{\widetilde{\mathrm{Ric}}\,(N, N) + \alpha^2(n-2)\} = -(n-2)\,\delta\alpha.$$

Substituting for $\widetilde{\mathrm{Ric}}\,(N, N)$ from (4.8) we have

$$\epsilon\lambda\{a(n-2) + bf + \alpha^2(n-2)\} = -(n-2)\,\delta\alpha. \qquad \ldots(4.8)$$

Equations (4.6) and (4.8) yield $b = 0$ and hence

$$\delta\alpha = -\epsilon\lambda(a + \alpha^2).$$

As a result we have the following:

Corollary 4.4 — If the infinitesimal normal parallel variation carries a normal paracontact hypersurface (with f-sectional curvature $-\alpha^2$) of a manifold of almost constant curvature to a normal paracontact hypersurface with f-sectional curvature $-\alpha^2 - \delta\alpha^2$, then

$$\delta\alpha = -\epsilon\lambda(a + \alpha^2)$$

and the enveloping manifold reduces to an Einstein manifold.

Corollary 4.5 — If the infinitesimal normal parallel variation carries a normal paracontact hypersurface (with f-sectional curvature $-\alpha^2$) of a manifold of almost constant curvature to a normal paracontact hypersurface (with f-sectional curvature $-\alpha^2$), then the f-sectional curvature of either of the original and varied hypersurfaces is equal to the almost constant curvature a of the enveloping manifold which reduces to an Einstein manifold whose scalar curvature is equal to $n(n-2)\,a$.

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