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SOME SPECIAL PROPERTIES OF R^h_{ijk} AND Q^h_{ijk} IN RECURRENT FINSLER SPACES

H. D. PANDE

Department of Mathematics, University of Jabalpur, Jabalpur (M.P.)

AND

K. K. GUPTA

Department of Mathematics, U.N.P.G. College, Padrauna, Dist. Deoria (U.P.)

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The purpose of present paper is to obtain some important identities under decomposition of curvature tensors R^h_{ijk} and Q^h_{ijk} in $R-\oplus$ recurrent and $Q-\oplus$ recurrent Finsler spaces F^*_n (Rund 1959) equipped with non-symmetric connection parameters.

1. INTRODUCTION

Let F^*_n denote an n-dimensional Finsler space equipped with $2n$-line-elements $(x^i, \dot{x}^i)$, $(i, j, \ldots = 1, 2, 3, \ldots, n)$ and a non-symmetric connection parameter

$$\Gamma^i_{jk}(x, \dot{x}) \neq (\Gamma^i_{kj}(x, \dot{x}))$$

that is based on non-symmetric fundamental metric tensor $g_{ij}(x, \dot{x}) \neq g_{ji}(x, \dot{x})$. Following Pande and Gupta (1978), we assume that Γ^i_{jk} are homogeneous of degree zero in its directional arguments $\dot{x}^i$'s.

$$\Gamma^i_{jk} = M^i_{jk} + \tfrac{1}{2} N^i_{jk} \qquad \ldots(1.1)$$

where M^i_{jk} and $\frac{1}{2} N^i_{jk}$ denote symmetric and skew-symmetric parts of Γ^i_{jk} ·

The covariant derivatives of tensor field $T^i_j(x, \dot{x})$ with respect to x^k are defined in two ways:

$$T^{\overset{i}{+}}_{\underset{+}{j}|k} = \partial_k T^i_j - (\dot{\partial}_m T^i_j)\dagger\, \Gamma^m_{pk}\dot{x}^p + T^m_j \Gamma^i_{mk} - T^i_m \Gamma^m_{jk} \qquad \ldots(1.2)$$

† $\partial_i \equiv \partial/\partial x^i$, $\dot{\partial}_i \equiv \partial/\partial \dot{x}^i$.

$$T^{\ i}_{\underset{-}{j}|k} = \partial_k T^i_j - (\dot{\partial}_m T^i_j)\,\tilde{\Gamma}^m_{pk}\,\dot{x}^p + T^m_j\,\tilde{\Gamma}^i_{mk} - T^i_m\,\tilde{\Gamma}^m_{jk} \qquad \text{...(1.3)}$$

where $\tilde{\Gamma}^i_{jk} \equiv \Gamma^i_{kj}$ and positive, negative, signs have their specific meanings as mentioned in Pande and Gupta (1977) and Cātālina Nitescu (1974). The commutation formulae, involving the co-variant derivatives defined in (1.2) and (1.3) give rise to two curvature tensors.

$$R^i_{jkl} = \partial_l \Gamma^i_{jk} - \partial_k \Gamma^i_{jl} - (\dot{\partial}_m \Gamma^i_{jk})\,\Gamma^m_{pl}\,\dot{x}^p + (\dot{\partial}_m \Gamma^i_{jl})\,\Gamma^m_{pk}\,\dot{x}^p + \Gamma^p_{jk}\,\Gamma^i_{pl} - \Gamma^p_{jl}\,\Gamma^i_{pk}\,. \qquad \text{...(1.4)}$$

and

$$\tilde{R}^i_{jkl} = \partial_l \tilde{\Gamma}^i_{jk} - \partial_k \tilde{\Gamma}^i_{jl} - (\dot{\partial}_m \tilde{\Gamma}^i_{jk})\,\tilde{\Gamma}^m_{pl}\,\dot{x}^p + (\dot{\partial}_m \tilde{\Gamma}^i_{jl})\,\tilde{\Gamma}^m_{pk}\,\dot{x}^p + \tilde{\Gamma}^p_{jk}\,\tilde{\Gamma}^i_{pl} - \tilde{\Gamma}^p_{jl}\,\Gamma^i_{pk}\,. \qquad \text{...(1.5)}$$

The following commutation formulae and identities have been obtained in Pande and Gupta (1977, 1978) :

$$\dot{\partial}_k\left(T^{\ i}_{\underset{+}{j}|h}\right) - \left(\dot{\partial}_{\underset{+}{k}} T^{\ i}_{\underset{+}{j}}\right)_{|h} = T^m_j\,\dot{\partial}_k \Gamma^i_{mh} - T^i_m\,\dot{\partial}_k \Gamma^m_{jh} - (\dot{\partial}_m T^i_j)\,(\dot{\partial}_k \Gamma^m_{ph})\,\dot{x}^p \qquad \text{...(1.6)}$$

$$T^{\ i}_{\underset{+}{j}|hk} - T^{\ i}_{\underset{+}{j}|kh} = -\,(\dot{\partial}_m T^i_j)\,R^m_{hk} + T^m_j\,R^i_{mhk} - T^i_m\,R^m_{jhk} + \left(T^{\ i}_{\underset{+}{j}|m}\right) N^m_{kh} \qquad \text{...(1.7)}$$

where

$$N^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj} \qquad \text{...(1.8)}$$

$$R^i_{hjk} = -\,R^i_{hkj},\ R^i_{jk} = -\,R^i_{kj},\ N^i_{jk} = -\,N^i_{kj} \qquad \text{...(1.9)}$$

$$x^{+\,i}_{\ |k} = x^{-\,i}_{\ |k} = 0 \qquad \text{...(1.10)}$$

$$R^i_l \overset{def}{\equiv} \dot{x}^k R^i_{kl} \overset{def}{\equiv} \dot{x}^k \dot{x}^j R^i_{jkl} \qquad \text{...(1.11)}$$

and

$$R \overset{def}{\equiv} \frac{1}{n-1} R^i_i \, , \; R_{jk} \overset{def}{\equiv} R^i_{jki}. \qquad \ldots(1.12)$$

The projective covariant derivative of $T^i_j(x, \dot{x})$ with respect to x^k is by Pande and Gupta (1977)

$$T^{\overset{i}{+}}_{\underset{+}{j}\,||\,k} = \partial_k T^i_j - (\dot{\partial}_m T^i_j)\, \pi^m_{pk}\, \dot{x}^p + T^m_j\, \pi^i_{mk} - T^i_m\, \pi^m_{jk} \qquad \ldots(1.13)$$

where

$$\pi^i_{jk} \overset{def}{=} \Gamma^i_{jk} - \frac{1}{n+1}\,(\delta^i_j\, \Gamma^r_{rk} + \delta^i_k\, \Gamma^r_{rj} + \dot{x}^i \dot{\partial}_j \Gamma^r_{rk}) \qquad \ldots(1.14)$$

are called 'projective connection coefficients'. These are non-symmetric in their lower indices and are homogeneous of degree zero in directional arguments.

The duality in the nature of projective covariant derivatives defined in (1.13), introduces the following 'projective entities' Q^i_{jkl}, such that

$$\begin{aligned} Q^i_{jkl} = {} & R^i_{jkl} + \frac{1}{n+1}\,(\dot{x}^i \dot{\partial}_j R^r_{rlk} + \delta^i_j\, R^r_{rlk}) \\ & + \frac{2}{(n+1)^2}\, \{((n+1)\, \Gamma^r_{\underset{+}{rj}\,|[k}{}^{*} + A\dot{\partial}_{[k} \Gamma^r_{<rj>}{}^{**} \\ & + \Gamma^s_{s[k}\, \Gamma^r_{<rj>})\, \delta^i_{l]} + P^i_{jkl}\} \end{aligned} \qquad \ldots(1.15)$$

where

$$\begin{aligned} P^i_{jkl} \overset{def}{\equiv} {} & (n+1)\,(-\,\dot{x}^p \dot{x}^i \dot{\partial}_m \Gamma^r_{r[k}\, \dot{\partial}_{<j>}\, \Gamma^m_{<p>l]} - \tfrac{1}{2} N^i_{kl}\, \Gamma^r_{rj} \\ & - \dot{\partial}_{[k} \Gamma^i_{<j>l]}\, A) + \delta^i_j\, A\dot{\partial}_{[k} \Gamma^r_{<r>l]} + \dot{x}^i\, (A\dot{\partial}^2_{j[k}\, \Gamma^r_{<r>l]} \\ & + \Gamma^s_{sj}\, \dot{\partial}_{[k} \Gamma^r_{<r>l]}). \end{aligned} \qquad \ldots(1.16)$$

$$A \overset{def}{\equiv} \Gamma^r_{rj}\, \dot{x}^j, \; Q^i_l \overset{def}{\equiv} \dot{x}^k Q^i_{kl} \overset{def}{\equiv} \dot{x}^k \dot{x}^j Q^i_{jkl}, \; S^i_{jk} \overset{def}{\equiv} \pi^i_{jk} - \pi^i_{kj} \qquad \ldots(1.17)$$

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*$2T_{(hk)} = T_{hk} + T_{kh}$
$2T_{[hk]} = T_{hk} - T_{kh}$
$3T_{[hkl]} = T_{hkl} + T_{klh} + T_{lhk}$.

**The indices in brackets < > are free from symmetric and skew-symmetric parts.

$$Q_{jk} \overset{def}{\equiv} Q^{i}_{jki},\ Q \overset{def}{\equiv} \frac{1}{n-1} Q^{i}_{i},\ Q^{i}_{jkl} = -Q^{i}_{jlk},\ Q^{i}_{jk} = -Q^{i}_{kj} \quad \ldots(1.18)$$

$$P^{i}_{l} \overset{def}{\equiv} \dot{x}^{k} P^{i}_{kl} \overset{def}{\equiv} \dot{x}^{k}\dot{x}^{j} P^{i}_{jkl},\ P_{jk} \overset{def}{=} P^{i}_{jki},\ P \overset{def}{=} \frac{1}{n-1} P^{i}_{i}. \quad \ldots(1.19)$$

The following identities are also satisfied by R^{i}_{jkl} and Q^{i}_{jkl} as (Pande and Gupta 1977, 1978)

$$R^{i}_{j(kh)} = 0 = Q^{i}_{j(kh)},\ Q^{i}_{ikl} = 0,\ \pi^{i}_{ik} = 0 \quad \ldots(1.20)$$

$$R^{i}_{[jkh]} = N^{\overset{i}{+}}_{\underset{++}{[j\,k}|h]};\ \dot{x}^{\overset{i}{+}}_{\|k} = 0 = \dot{x}^{\overset{i}{-}}_{\|k} \quad \ldots(1.21)$$

$$R^{\overset{h}{+}}_{\underset{+}{i}[jk|l]} + \dot{\partial}_{m}\Gamma^{h}_{i[l} R^{m}_{jk]} = 0 \quad \ldots(1.22)$$

$$R^{\overset{i}{+}}_{\underset{+}{l}|h} - R^{\overset{i}{+}}_{\underset{+}{h}|l} + R^{\overset{i}{+}}_{\underset{++}{l\,h}|k} \dot{x}^{k} = -3\dot{x}^{j}\dot{x}^{k}\{\dot{\partial}_{m}\Gamma^{i}_{j[h} R^{m}_{kl]} + R^{i}_{jm[l} N^{m}_{kh]}\} \quad \ldots(1.23)$$

$$\begin{aligned} Q^{i}_{[jkl]} = R^{i}_{[jkl]} + \frac{1}{n+1} \Big\{ & \dot{x}^{i}\beta_{[jk|l]} + A\dot{\partial}_{[l}N^{i}_{jk]} \\ & - \Gamma^{r}_{rp} N^{p}_{[jk}\delta^{i}_{l]} - \Gamma^{r}_{r[l} N^{i}_{jk]} - \dot{x}^{i}\dot{\partial}_{p}\Gamma^{r}_{r[l} N^{p}_{jk]} \\ & + \frac{\dot{x}^{i}}{n+1}(A\dot{\partial}_{[l}\beta_{jk]} - \Gamma^{r}_{r[l}\beta_{jk]})\Big\} \end{aligned} \quad \ldots(1.24)$$

where

$$\beta_{ij} \overset{def}{=} \dot{\partial}_{j}\Gamma^{r}_{ri} - \dot{\partial}_{i}\Gamma^{r}_{rj} \quad \ldots(1.25)$$

$$\begin{aligned} Q^{\overset{i}{+}}_{\underset{+}{l}|h} - Q^{\overset{i}{+}}_{\underset{+}{h}|l} + Q^{\overset{i}{+}}_{\underset{++}{l\,h}|k}\dot{x}^{k} = & -3\dot{x}^{k}(\dot{x}^{j}\dot{\partial}_{m}\Gamma^{i}_{j[h} Q^{m}_{kl]} \\ & + Q^{i}_{m[l} N^{m}_{kh]}) + \frac{1}{n+1}\{Q^{m}_{l}\delta^{i}_{h}(\dot{\partial}_{m}A) \\ & + Q^{m}_{lh}\dot{x}^{i}(2\dot{\partial}_{m}A - \Gamma^{r}_{rm}) - Q^{m}_{h}\delta^{i}_{l}\dot{\partial}_{m}A + \dot{x}^{i}Q^{p}_{l}\dot{\partial}_{p}\Gamma^{r}_{rh} \\ & - \dot{x}^{i}Q^{p}_{h}\dot{\partial}_{p}\Gamma^{r}_{rl} + A\dot{x}^{k}(\dot{\partial}_{l}Q^{i}_{kh} - \dot{\partial}_{h}Q^{i}_{kl}) - AQ^{i}_{lh}\}. \end{aligned} \quad \ldots(1.26)$$

$$\underset{+}{\overset{h}{Q^{+}}}_{i[jk|l]} = \underset{+}{\overset{h}{R^{+}}}_{i[jk|l]} + \frac{2}{(n+1)^2}\underset{+}{\overset{h}{P^{+}}}_{i[jk|l]} + \frac{1}{n+1}(\dot{\partial}_i(\dot{x}^h R^r_{r[kj|l]})$$

$$+ \dot{x}^h \dot{\partial}_m R^r_{r[kj} \dot{\partial}_{<i>} \Gamma^m_{<p>l]} \dot{x}^p) + \frac{1}{(n+1)^2}\{((n+1)\Gamma^r_{r\underset{+}{i}}|_{[jl}$$

$$+ A_{|[l} \dot{\partial}_j \Gamma^r_{<ri>} + A(\dot{\partial}_{[j}\Gamma^r_{<r\underset{+}{i}>})_{|l}$$

$$+ \Gamma^s_{s[j|l}\Gamma^r_{<ri>} + \Gamma^s_{s[j}\Gamma^r_{<r\underset{+}{i}>|l}) \delta^h_{k]}$$

$$- ((n+1)\Gamma^r_{r\underset{+}{i}}|_{[kl} + A_{|[l}\dot{\partial}_k\Gamma^r_{<ri>}$$

$$+ A(\dot{\partial}_{[k}\Gamma^r_{<r\underset{+}{i}>})_{|l} + \Gamma^r_{ri}\Gamma^s_{s[k|l}$$

$$+ \Gamma^s_{s[k}\Gamma^r_{<r\underset{+}{i}>|l}) \delta^h_{j]}\}. \qquad \ldots(1.27)$$

$$\overset{h}{Q^{+}}_{[jk|l]} - \frac{1}{n+1}\dot{x}^h R^r_{r[kj|l]} = -\dot{x}^i \dot{\partial}_m \Gamma^h_{i[l} R^m_{jk]}$$

$$+ \frac{1}{(n+1)^2}\{(n+1)\delta^h_{[k} A_{|jl]}$$

$$+ (A\dot{\partial}_{[j}A)_{|l}\delta^h_{k]} - (n+1)\delta^h_{[j} A_{|kl]}$$

$$- (A\dot{\partial}_{[k}A)_{|l}\delta^h_{j]} + 2\overset{h}{P^{+}}_{[jk|l]}\}. \qquad \ldots(1.28)$$

$$Q^i_{jkh} = (R^i_{jkh} + \frac{2}{(n+1)^2}P^i_{jkh}) + \frac{1}{n+1}(\dot{x}^i\dot{\partial}_j R^r_{rhk}$$

$$+ \delta^i_j R^r_{rhk}) + \frac{2}{n-1}\Big\{Q_{j[k} - R_{j[k} - \frac{2}{(n+1)^2}P_{j[k}$$

$$- \frac{1}{n+1}(\dot{x}^p\dot{\partial}_j R^r_{rp[k} + R^r_{rj[k})\Big\}\delta^i_{h]}\,. \qquad \ldots(1.29)$$

2. $R - \oplus$ Recurrent Finsler Space

Definition 2.1 — An n-dimensional Finsler space F^*_n will be called $R - \oplus$ recurrent F^*_n if

$$\overset{i}{R^{+}}_{\underset{+++}{jkh}|l} = \lambda_l R^i_{jkh} \qquad \ldots(2.1)$$

where $\lambda_l \equiv \lambda_l(x)$ is a non-zero recurrence vecotor field and R^i_{jkh} is non-zero curvature tensor of the space.

Transvecting (2.1) with $\dot{x}^j$ and $\dot{x}^k$ successively and using (1.11), (1.10), we get

$$R^{\overset{i}{+}}_{\underset{++}{kh}|l} = \lambda_l R^i_{kh} \quad \text{...(2.2)}$$

$$R^{\overset{i}{+}}_{\underset{+}{h}|l} = \lambda_l R^i_h . \quad \text{...(2.3)}$$

Contracting i and h in (2.3) and using (1.12), we have

$$R_{|l} = \lambda_l R \quad \text{...(2.4)}$$

and hence

$$\lambda_l = (\log R)_{|l}. \quad \text{...(2.5)}$$

In this section, we shall obtain some identities under the condition, when R^i_{jkh} is expressed in the following form in a $R - \oplus$ recurrent F^*_n.

$$R^i_{jkh} = X^i_j \, \phi_{kh} \quad \text{...(2.6)}$$

and

$$X^i_j \, \lambda_i = a_j \quad \text{...(2.7)}$$

where X^i_j and ϕ_{kh} are non-zero tensor fields and λ_i, a_j are non-zero covariant recurrence and decompose vector fields respectively.

Theorem 2.1 — In view of (2.6) and (2.7), the identities for R^i_{jkh} in a $R - \oplus$ recurrent F^*_n are given by

$$a_j \phi_{(kh)} = 0 \quad \text{...(2.8)}$$

$$a_{[j} \phi_{kh]} = \lambda_i N^{\overset{i}{+}}_{\underset{++}{[jk}|h]} \quad \text{...(2.9)}$$

$$a_i(\lambda_{[l} \phi_{jk]} + \phi_{m[k} N^m_{jl]}) + X^m_p \, \dot{x}^p \dot{\partial}_m(\lambda_h \Gamma^h_{i[l}) \, \phi_{jk]} = 0 \quad \text{...(2.10)}$$

$$a_i \lambda_l R^l_{hjk} = a_i a_h \phi_{jk} = a_i(\lambda_j S_{hk} - \lambda_k S_{hj}) - X^l_h \, (a_i \phi_{m[k} N^m_{jl]} + X^m_p \, \dot{x}^p \dot{\partial}_m(\lambda_h \Gamma^h_{i[l}) \, \phi_{jk]}) \quad \text{...(2.11)}$$

where

$$S_{li} \overset{def}{=} X_l^h \phi_{hi}. \qquad ...(2.12)$$

PROOF : Using (1.20) and (2.6), we have

$$X_j^i \phi_{(kh)} = 0.$$

Transvecting it with λ_i and using (2.7) we get (2.8).

In view of (1.21) and (2.6), we get

$$X^i_{[j} \phi_{kh]} = N^{\overset{i}{+}}_{\underset{++}{[jk}|h]} .$$

A transvection of above with λ_i gives (2.9), in view of (2.7).

The Bianchi's identity (1.22) can be rewritten as

$$R^{\overset{h}{+}}_{\underset{+\ ++}{i[jk}|l]} + \dot{\partial}_m \Gamma^h_{i[l} R^m_{jk]} + R^h_{im[k} \Gamma^m_{jl]} - R^h_{im[j} \Gamma^m_{kl]} = 0 \qquad ...(2.13)$$

in view of (1.9) and specific meaning of positive signs (Pande and Gupta 1977).

Using (2.6) and (2.1) in (2.13) and transvecting the obtained expression with λ_h, we have (2.10) in view of (2.7).

In view of (2.7) and (2.6), we have

$$a_i a_h \phi_{jk} = a_i \lambda_l X^l_h \phi_{jk} = a_i \lambda_l R^l_{hjk}. \qquad ...(2.14)$$

A transvection of (2.10) with X^l_h gives (2.11) in view of (2.12) and (2.14).

Theorem 2.2 — The tensor field ϕ_{kh} is recurrent, if X^i_j is covariantly invariant in a $R - \oplus$ recurrent F^*_n.

PROOF : A covariant differentiation of (2.6) with respet to x^l yields

$$X^i_j (\phi_{\underset{++}{kh}|l} - \lambda_l \phi_{kh}) = 0 \qquad ...(2.15)$$

in view of (2.1), (2.6) and the hypothesis that X^i_j is covariantly invariant, viz.

$$X^{\overset{i}{+}}_{\underset{+}{j}|l} = 0.$$

Now, X^i_j being non-zero tensor field, we have

$$\phi_{\underset{++}{kh}|l} = \lambda_l \phi_{kh}$$

and this establishes the theorem (2.2).

3. $Q - \oplus$ Recurrent Finsler Space

Definition 3.1 — F^*_n will be called $Q - \oplus$ recurrent, if the projective entities satisfy the relation

$$Q^{\overset{h}{+}}_{\underset{+++}{ijk}|l} = \lambda_l Q^h_{ijk}, \ (Q^h_{ijk} \neq 0) \qquad \ldots(3.1)$$

where

$$\lambda_l \equiv \lambda_l(x) \qquad \ldots(3.2)$$

is non-zero recurrence vector field, independent of directional arguments.

Transvections of (3.1) with $\dot{x}^i$ and $\dot{x}^j$ successively give the following results, in view of (1.17) and (1.21).

$$Q^{\overset{h}{+}}_{\underset{++}{jk}|l} = \lambda_l Q^h_{jk} \qquad \ldots(3.3)$$

$$Q^{\overset{h}{+}}_{\underset{+}{k}|l} = \lambda_l Q^h_k. \qquad \ldots(3.4)$$

A contraction of indices h and k in (3.4) gives

$$Q_{|l} = \lambda_l Q \qquad \ldots(3.5)$$

in view of (1.18).

In this section, some identities will be obtained, when Q^h_{ijk} takes the following particular form in a $Q - \oplus$ recurrent F^*_n.

$$Q^h_{ijk} = \gamma^h_i \, \psi_{jk} \qquad \ldots(3.6)$$

where γ^h_i and ψ_{jk} are non-zero tensor-fields such that

$$\gamma^h_i \lambda_h = b_i \qquad \ldots(3.7)$$

b_i will be called decompose vector-field under (3.6).

Theorem 3.1 — In view of (3.6), projective entities Q^{i}_{jkl} satisfy the following identities

$$b_{[j}\psi_{kl]} = a_{[j}\phi_{kl]} + \frac{1}{n+1}\Big\{\lambda_i \dot{x}^i(\beta_{[jk\,|\,l]} - \dot{\partial}_p \Gamma^{r}_{r[l} N^{p}_{jk]} + \frac{1}{n+1}(A\dot{\partial}_{[l}\beta_{jk]} - \Gamma^{r}_{r[l}\,\beta_{jk]}) + \lambda_i(A\dot{\partial}_{[l}N^{i}_{jk]} - \Gamma^{r}_{r[l}\,N^{i}_{jk]}) - \Gamma^{r}_{rp}\,N^{p}_{[jk}\lambda_{l]}\Big\} \quad \ldots(3.8)$$

$$b_p\dot{x}^p\left(\lambda_h\dot{x}^q\psi_{ql} - \lambda_l\dot{x}^q\psi_{qh} + \lambda_k\psi_{lh}\dot{x}^k + 3\psi_{m[l}N^{m}_{kh]}\,\dot{x}^k + \frac{A}{n+1}\psi_{lh}\right)$$

$$= \gamma^{m}_{p}\,\dot{x}^p\Big\{-3\dot{x}^k\dot{x}^j\dot{\partial}_m(\lambda_i\Gamma^{i}_{j[h})\,\psi_{kl]} + \frac{1}{n+1}(\dot{x}^q\psi_{ql}\lambda_h\dot{\partial}_m A + \lambda_i\dot{x}^i\{\psi_{lh}(2\dot{\partial}_m A - \Gamma^{r}_{rm}) + \dot{x}^q\psi_{ql}\dot{\partial}_m\Gamma^{r}_{rh} - \dot{x}^q\psi_{qh}\dot{\partial}_m\Gamma^{r}_{rl}\}$$

$$- \dot{x}^q\psi_{qh}\lambda_l\dot{\partial}_m A)\Big\} + \frac{A\dot{x}^k}{n+1}\{\dot{\partial}_l(b_p\dot{x}^p\psi_{kh}) - \dot{\partial}_h(b_p\dot{x}^p\psi_{kl})\} \quad \ldots(3.9)$$

$$b_p\dot{x}^p(\lambda_{[l}\psi_{jk]} + \psi_{m[k}\Gamma^{m}_{jl]} - \psi_{m[j}\Gamma^{m}_{kl]}) + \dot{x}^i X^{m}_{p}\,\dot{x}^p\dot{\partial}_m(\lambda_h\Gamma^{h}_{i[l})\,\phi_{jk]}$$

$$- X^{r}_{r}\,\lambda_h\dot{x}^h\,\frac{1}{n+1}\{\lambda_{[l}\phi_{kj]} + \phi_{m[j}\Gamma^{m}_{kl]} - \phi_{m[k}\Gamma^{m}_{jl]}\}$$

$$= \frac{1}{(n+1)^2}\{(n+1)\,(\lambda_{[k}A_{\,|\,jl]} - \lambda_{[j}A_{\,|\,kl]}) + (A\dot{\partial}_{[j}A)_{\,|\,l}\lambda_{k]} - (A\dot{\partial}_{[k}A)_{\,|\,l}\lambda_{j]} + 2\lambda_h \overset{h}{\underset{}{P}}{}^{+}_{[jk\,|\,l]}\}. \quad \ldots(3.10)$$

$$\lambda_l a_j\phi_{kh} + \frac{1}{n+1}[\lambda_i\dot{x}^i\{\lambda_l\dot{\partial}_j(X^{r}_{r}\phi_{hk}) + \dot{\partial}_m(X^{r}_{r}\phi_{hk})\,\dot{\partial}_j\Gamma^{m}_{pl}\,\dot{x}^p + X^{r}_{r}(\phi_{mk}\dot{\partial}_j\Gamma^{m}_{hl} + \phi_{hm}\dot{\partial}_j\Gamma^{m}_{kl})\} + \lambda_l X^{r}_{r}\,\lambda_j\phi_{hk}] + \frac{2\lambda_i}{(n+1)^2}\overset{i}{\underset{+++}{P}}{}_{jkh\,|\,l}$$

$$+ \frac{2}{n-1}\Big[\lambda_l(X^{p}_{j}\,\phi_{p[k} - \gamma^{p}_{j}\psi_{p[k}) - \frac{2}{(n+1)^2}\underset{++}{P}{}_{j\,[k\,|\,<l>} - \frac{1}{n+1}(\{X^{r}_{r}\,(\dot{\partial}_j\Gamma^{m}_{pl}\,\phi_{m[k} + \phi_{pm}\dot{\partial}_j\Gamma^{m}_{[k<l>}) + \lambda_l\dot{\partial}_j(X^{r}_{r}\phi_{p[k})$$

$$+ \dot{\partial}_j\Gamma^{m}_{ql}\,\dot{x}^q\dot{\partial}_m(X^{r}_{r}\,\phi_{p[k})\}\,\dot{x}^p + \lambda_l X^{r}_{r}\,\phi_{j[k)}\Big]\lambda_{h]} = \lambda_l b_j\psi_{kh} \quad \ldots(3.11)$$

PROOF : Transvecting the expression obtained by using (3.6) in (1.24), with λ_i we have (3.8) in view of (3.7) and (2.7).

Using (3.3), (3.4), (3.6), (1.17) in (1.26) and then transvecting the expression obtained with λ_i, we obtain (3.9), in view of (2.7), (3.7) and (3.2).

In view of (1.2), (1.18), it is easily seen that

$$\overset{h}{\underset{}{Q}}{}^{+}_{[jk|l]} = \overset{h}{Q}{}^{+}_{[\underset{++}{j\,k}|l]} + Q^h_{m[k}\Gamma^m_{jl]} - Q^h_{m[j}\Gamma^m_{kl]} \qquad \text{...(3.12)}$$

$$R^r_{r[kj|l]} = \overset{r}{R}{}^{+}_{\underset{+\,++}{r\,[k\,j}|l]} + R^r_{rm[j}\Gamma^m_{kl]} - R^r_{rm[k}\Gamma^m_{jl]} \qquad \text{...(3.13)}$$

Using (3.12), (3.13), (3.1), (2.1), (1.11), (3.6) successively in (1.28) and then transvecting the expression obtained with λ_h we get (3.10) in view of (3.7) and (3.2), (2.7).

In view of (1.6), (3.6), (2.1), we have

$$(\dot\partial_{\underset{+}{j}} \overset{r}{R}{}^{+}_{\underset{+++}{r\,h\,k}})_{|l} = \lambda_l\dot\partial_j(X^r_r\,\phi_{hk}) + X^r_r\,(\phi_{mk}\dot\partial_j\Gamma^m_{hl} + \phi_{hm}\dot\partial_j\Gamma^m_{kl})$$

$$+ \dot\partial_m(X^r_r\,\phi_{hk})\,(\dot\partial_j\Gamma^m_{pl})\,\dot x^p. \qquad \text{...(3.14)}$$

Similarly,

$$(\dot\partial_j \overset{r}{R}{}^{+}_{\underset{+++}{r\,p\,k}})_{|l} = \lambda_l\dot\partial_j(X^r_r\,\phi_{pk}) + X^r_r\,(\phi_{mk}\dot\partial_j\Gamma^m_{pl} + \phi_{pm}\dot\partial_j\Gamma^m_{kl})$$

$$+ \dot\partial_m(X^r_r\,\phi_{pk})\,\dot\partial_j\Gamma^m_{ql}\,\dot x^q \qquad \text{...(3.15)}$$

and with the help of (1.18), (1.12), (2.1), (3.1), (3.6) and (2.6) we have

$$Q_{\underset{++}{j\,k}|l} = \lambda_l\Upsilon^p_j\,\psi_{kp} \qquad \text{...(3.16)}$$

$$R_{\underset{++}{j\,k}|l} = \lambda_l X^p_j\,\phi_{kp}. \qquad \text{...(3.17)}$$

Differentiating (1.29) covariantly with respect to x^l, we have

$$\overset{i}{Q}{}^{+}_{\underset{+++}{j\,k\,h}|l} = \overset{i}{R}{}^{+}_{\underset{+++}{j\,k\,h}|l} + \frac{1}{n+1}\,(\dot x^i(\dot\partial_{\underset{+}{j}}\,\overset{r}{R}{}^{+}_{\underset{+++}{r\,h\,k}})_{|l} + \delta^i_j\,(\overset{r}{R}{}^{+}_{\underset{+++}{r\,h\,k}})_{|l})$$

$$+ \frac{2}{(n+1)^2}\,\overset{i}{P}{}^{+}_{\underset{+++}{j\,k\,h}|l} + \frac{2}{n-1}\Big\{Q_{\underset{++}{j\,[k}|<l>} - R_{\underset{++}{j\,[k}<l>}$$

$$- \frac{2}{(n+1)^2}\,P_{\underset{++}{j\,[k}|<l>} - \frac{1}{n+1}\,(\dot x^p(\dot\partial_{\underset{+}{j}}\overset{r}{R}{}^{+}_{\underset{+++}{r\,p\,[k}})_{|<l>}$$

$$+ \overset{r}{R}{}^{+}_{\underset{+++}{r\,j\,[k}|<l>})\Big\}\,\delta^i_{h]}\,. \qquad \text{...(3.18)}$$

Using (3.14), (3.15), (3.16), (3.17), (2.1), (2.6) in (3.18) and then transvecting the expression obtained with λ_i we have (3.11) in view of (2.7), (3.7) and (3.2).

The identities parallel to above can be obtained in $\bar{R} - \ominus$ recurrent and $\tilde{Q} - \ominus$ recurrent Finsler spaces after replacing Γ^i_{jk} by $\tilde{\Gamma}^i_{jk}$.

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