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ON CONFORMAL TRANSFORMATION IN AREAL SPACE

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In one of the recent papers Prasad and Bose (1979) have introduced the notion of the conformal transformation in an areal space. Pande and Dwivedi (1979) have obtained the covariant derivatives of $g^{\alpha\beta}_{ij}$[1] and $C^{\alpha\beta\gamma}_{ijk}$ in conformally transformed Areal space. The purpose of the present paper is to find conformal transformation of $\underset{1}{K}{}^{i}_{jkh}$ and $\underset{2}{K}{}^{i}_{jkh}$ and the covariant derivatives of $g^{ij}_{\alpha\beta}$ and $C^{ijk}_{\alpha\beta\gamma}$ in conformally transformed Areal space.

1. INTRODUCTION

In an n-dimensional Areal space $A^{(m)}_{n}$, the Areal metric tensor (Rund 1968) is defined as

$$g^{\alpha\beta}_{ij}\ (x^h, \dot{x}^h_\epsilon) = \left[\frac{m}{2}\ \frac{\partial^2 L^2/m}{\partial \dot{x}^i_\alpha\ \partial \dot{x}^j_\beta}\right]^{(2)} . \qquad \ldots(1.1)$$

where $L = L(x^h, \dot{x}^h_\epsilon)$ is a Lagrange function.

The areal metric tensor $g^{\alpha\beta}_{ij}$ and its inverse $g^{ij}_{\alpha\beta}$ are connected by

$$g^{\alpha\beta}_{ij}\ g^{ik}_{\alpha\gamma} = \delta^k_j\ \delta^\beta_\gamma \qquad \ldots(1.2)$$

where δ^k_j and δ^β_γ are Kronecker deltas.

Analogous to Finsler geometry, we also have

(1) Throughout this paper the Latin indices $i, j, k, \ldots$ run over 1 to n while Greek indices $\alpha, \beta, \gamma, \delta, \varepsilon \ldots$ run over 1 to m.

(2) $\dot{x}^i_\alpha = \dfrac{\partial x^i}{\partial t^\alpha}$.

$$C^{\alpha\beta\gamma}_{ijk} = \tfrac{1}{2} \frac{\partial g^{\alpha\beta}_{ij}}{\partial \dot{x}^k_\gamma} \qquad \text{...(1.3)}$$

which is symmetric in pairs of indices such as (α, i), (β, j) and (γ, k).

The curvature tensors of first and second kinds (Rund 1968) are given by

$$\underset{1}{K}{}^i_{jkh} = \left(\frac{\partial \Gamma^i_{jk}}{\partial x^h} - \frac{\partial \Gamma^i_{jk}}{\partial \dot{x}^l_\alpha} \Gamma^l_{ph} \dot{x}^p_\alpha\right) - \left(\frac{\partial \Gamma^i_{jh}}{\partial x^k} - \frac{\partial \Gamma^i_{jh}}{\partial \dot{x}^1_\alpha} \Gamma^i_{pk} \dot{x}^p_\alpha\right)$$

$$+ \Gamma^i_{ph} \Gamma^p_{jk} - \Gamma^i_{pk} \Gamma^p_{jh} \qquad \text{...(1.4)}$$

and

$$\underset{2}{K}{}^i_{jkh} = \left(\frac{\partial \Gamma^i_{kj}}{\partial x^h} - \frac{\partial \Gamma^i_{kj}}{\partial \dot{x}^l_\alpha} \Gamma^l_{ph} \dot{x}^p_\alpha\right) - \left(\frac{\partial \Gamma^i_{hj}}{\partial x^k} - \frac{\partial \Gamma^i_{hj}}{\partial \dot{x}^l_\alpha} \Gamma^l_{pk} \dot{x}^p_\alpha\right)$$

$$+ \Gamma^i_{hp} \Gamma^p_{kj} - \Gamma^i_{kp} \Gamma^p_{hj} \qquad \text{...(1.5)}$$

respectively, where Γ^i_{kj} is a connection coefficient.

The covariant derivative (Rund 1968) of a vector field $X^i_\epsilon (x^h, \dot{x}^h_\epsilon)$ with respect to x^j is given by

$$X^i_{\epsilon|j} = \frac{\partial \Gamma^i_\epsilon}{\partial x^j} - \frac{\partial X^i_\epsilon}{\partial \dot{x}^l_\alpha} \Gamma^l_{pj} \dot{x}^p_\alpha + \Gamma^i_{jl} X^l_\epsilon \cdot \qquad \text{...(1.6)}$$

Since the connection coefficient Γ^i_{kj} is not in general symmetric in k and j so second type of covariant derivative (Prasad and Bose 1976) of X^i_ϵ with respect to x^j is given by

$$X^i_{\epsilon\|j} = \frac{\partial X^i_\epsilon}{\partial x^j} - \frac{\partial X^i_\alpha}{\dot{x}^l_\alpha} \Gamma^l_{jp} \dot{x}^p_\alpha + \Gamma^i_{lj} X^l_\epsilon \cdot \qquad \text{...(1.7)}$$

2. Conformal Areal Space

Let two distinct Lagrange functions $L(x^h, \dot{x}^h_\epsilon)$ and $\bar{L}(x^h, \dot{x}^h_\epsilon)$ be defined over an n-dimensional space $A^{(m)}_n$ both of which satisfy the requisite conditions for an areal space. The two metrics resulting from these functions are called conformal if the corresponding areal metric tensors $g^{\alpha\beta}_{ij}$ and $\bar{g}^{\alpha\beta}_{ij}$ are proportional to each other. It has been proved (Prasad and Bose 1979) that the factor of proportionality between them is almost a point function.

Thus we have

$$\bar{g}^{\alpha\beta}_{ij} = e^{2\sigma} g^{\alpha\beta}_{ij} \qquad ...(2.1)$$

$$\bar{g}^{ij}_{\alpha\beta} = e^{-2\sigma} g^{ij}_{\alpha\beta} \qquad ...(2.2)$$

$$\bar{L} = e^{m\sigma} L \qquad ...(2.3)$$

where $\sigma = \sigma(x)$. The space $\bar{A}^{(m)}_{n}$ with the entities $\bar{L}$, $\bar{g}^{\alpha\beta}_{ij}$ etc. is called a conformal Areal space.

We shall denote the covariant derivatives given by (1.6) and (1.7) with respect to $\bar{g}^{\alpha\beta}_{ij}$ by putting a horizontal bar over the same notations of the covariant derivatives (i.e. $\bar{|}$ and $\bar{\|}$).

We shall use the following geometric entities of the conformal Areal space (Prasad and Bose 1979, Pande and Dwivedi 1979) :

$$\bar{C}^{\alpha\beta\gamma}_{ijk} = e^{2\sigma} C^{\alpha\beta\gamma}_{ijk} \qquad ...(2.4)$$

$$\bar{C}^{ijk}_{\alpha\beta\gamma} = e^{-2\sigma} C^{ijk}_{\alpha\beta\gamma} \qquad ...(2.5)$$

$$\bar{\Gamma}^{i}_{kj} = \Gamma^{i}_{kj} + U^{i}_{kj} \qquad ...(2.6a)$$

where

$$U^{i}_{kj} \overset{def}{=} A^{i\alpha}_{kj}\, x^{\cdot h}_{\alpha}\, \sigma_{,h} + B^{i}_{k}\, \sigma_{,j} + \frac{l}{m} G^{li}_{\varepsilon\beta}\, g^{\beta\varepsilon}_{lj}\, \sigma_{,k} - D^{il}_{kj}\, \sigma_{,l}. \qquad ...(2.6b)$$

The tensors $A^{i\alpha}_{kj}$, B^{i}_{k} and D^{il}_{kj} are conformally invariant.

$$\sigma_{,h} = \frac{\partial\sigma}{\partial x^{h}} \qquad ...(2.7)$$

$$\bar{G}^{\alpha\beta}_{kj} = e^{2\sigma} G^{\alpha\beta}_{kj} \qquad ...(2.8)$$

$$\bar{G}^{kj}_{\alpha\beta} = e^{-2\sigma} G^{kj}_{\alpha\beta} \qquad \text{...(2.9)}$$

$$\bar{\dot{x}}^{i}_{\alpha} = \dot{x}^{i}_{\alpha} \qquad \text{...(2.10)}$$

$$\bar{T}^{i}_{kh} = T^{i}_{kh} + 2U^{i}_{[kh]}{}^{*}. \qquad \text{...(2.11)}$$

3. Identities and Conformal Covariant Derivatives

With the help of (1.4), the curvature tensor of first kind in $\bar{A}^{(m)}_{n}$ becomes

$$\underset{1}{\bar{K}}{}^{i}_{jkh} = \left(\frac{\partial \bar{\Gamma}^{i}_{jk}}{\partial x^{h}} - \frac{\partial \bar{\Gamma}^{i}_{jk}}{\partial \dot{x}^{l}_{\alpha}} \bar{\Gamma}^{l}_{ph} \dot{x}^{p}_{\alpha}\right) - \left(\frac{\partial \bar{\Gamma}^{i}_{jh}}{\partial x^{k}} - \frac{\partial \bar{\Gamma}^{i}_{jh}}{\partial \dot{x}^{l}_{\alpha}} \bar{\Gamma}^{l}_{pk} \dot{x}^{p}_{\alpha}\right)$$

$$+ \bar{\Gamma}^{i}_{ph} \bar{\Gamma}^{p}_{jk} - \bar{\Gamma}^{i}_{pk} \bar{\Gamma}^{p}_{jh}. \qquad \text{...(3.1)}$$

Using the relation (2.6*a*) in (3.1) and then simplifying the result thus obtained in view of (1.4), we have

$$\underset{1}{\bar{K}}{}^{i}_{jkh} = \underset{1}{K}{}^{i}_{jkh} + \underset{1}{M}{}^{i}_{jkh} \qquad \text{...(3.2)}$$

where

$$\underset{1}{M}{}^{i}_{jkh} \overset{def}{=} \left(\frac{\partial U^{i}_{jh}}{\partial x^{h}} - \frac{\partial U^{i}_{jk}}{\partial \dot{x}^{l}_{\alpha}} U^{l}_{ph} \dot{x}^{p}_{\alpha}\right)$$

$$- \left(\frac{\partial U^{i}_{jh}}{\partial x^{k}} - \frac{\partial U^{i}_{jh}}{\partial \dot{x}^{l}_{\alpha}} U^{l}_{pk} \dot{x}^{p}_{\alpha}\right) + U^{i}_{ph} U^{p}_{jk} - U^{i}_{pk} U^{p}_{jh}$$

$$+ \frac{2\partial U^{i}_{j[h}}{\partial \dot{x}^{l}_{\alpha}} \Gamma^{l}_{<p>k}{}^{**} \dot{x}^{p}_{\alpha} + \frac{2\partial \Gamma^{i}_{j[h}}{\partial \dot{x}^{l}_{\alpha}} U^{l}_{<p>k]} \dot{x}^{p}_{\alpha}$$

$$+ 2U^{i}_{p[h} \Gamma^{p}_{<j>k]} + 2\Gamma^{i}_{p[h} U^{p}_{<j>k]}. \qquad \text{...(3.3)}$$

Again with the help of (1.5), the curvature tensor of second kind in $\bar{A}^{(m)}_{n}$ becomes

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$^{*}2U^{i}_{[kh]} = U^{i}_{kh} - U^{i}_{hk}$.

**The indices in $< >$ are free from symmetric and skew-symmetric parts.

$$\underset{2}{\bar{K}}{}^{i}_{jkh} = \left(\frac{\partial \bar{\Gamma}^{i}_{kj}}{\partial x^{h}} - \frac{\partial \bar{\Gamma}^{i}_{kj}}{\partial \dot{x}^{l}_{\alpha}} \bar{\Gamma}^{l}_{ph} \dot{x}^{p}_{\alpha}\right) - \left(\frac{\partial \bar{\Gamma}^{i}_{hj}}{\partial x^{k}} - \frac{\partial \bar{\Gamma}^{i}_{hj}}{\partial \dot{x}^{l}_{\alpha}} \bar{\Gamma}^{l}_{pk} \dot{x}^{p}_{\alpha}\right)$$

$$+ \bar{\Gamma}^{i}_{hp} \bar{\Gamma}^{p}_{kj} - \bar{\Gamma}^{i}_{kp} \bar{\Gamma}^{p}_{hj}. \qquad \ldots(3.4)$$

Using the relation (2.6*a*) in (3.4) and then simplifying the result thus obtained in view of (1.5), we get

$$\underset{2}{\bar{K}}{}^{i}_{jkh} = \underset{2}{K}{}^{i}_{jkh} + \underset{2}{M}{}^{i}_{jkh} \qquad \ldots(3.5)$$

where

$$\underset{2}{M}{}^{i}_{jkh} \overset{def}{=} \left(\frac{\partial U^{i}_{kj}}{\partial x^{h}} - \frac{\partial U^{i}_{kj}}{\partial \dot{x}^{l}_{\alpha}} U^{l}_{ph} \dot{x}^{p}_{\alpha}\right) - \left(\frac{\partial U^{i}_{hj}}{\partial x^{k}} - \frac{\partial U^{i}_{hj}}{\partial \dot{x}^{l}_{\alpha}} U^{l}_{pk} \dot{x}^{p}_{\alpha}\right)$$

$$+ U^{i}_{hp} U^{p}_{kj} - U^{i}_{kp} U^{p}_{hj} + \frac{2\partial U^{i}_{[h<j>}}{\partial \dot{x}^{l}_{\alpha}} \Gamma^{i}_{<p>k]} \dot{x}^{p}_{\alpha}$$

$$+ \frac{2\partial \Gamma^{i}_{[h<j>}}{\partial \dot{x}^{l}_{\alpha}} U^{l}_{<p>k]} \dot{x}^{p}_{\alpha} + 2U^{i}_{[h<p>} \Gamma^{p}_{k]j}$$

$$+ 2\Gamma^{i}_{[h<p>} U^{p}_{k]j}. \qquad \ldots(3.6)$$

On summarizing all the above results, we get the following theorems:

Theorem 3.1 — The conformal transformation of the curvature tensor of first kind is given by (3.2).

Theorem 3.2 — The conformal transformation of the curvature tensor of second kind is given by (3.5).

Corollary 3.1 — Under homothetic transformation (σ = constant) the curvature tensor of first kind is invariant.

Corollary 3.2 — Under homothetic transformation the curvature tensor of second kind is invariant.

Theorem 3.3 — If $A^{(m)}_{n}$ and $\bar{A}^{(m)}_{n}$ are in conformal correspondence, the covariant derivatives of $\bar{g}^{ij}_{\alpha\beta}$ and $\bar{C}^{ijk}_{\alpha\beta\gamma}$ are given by

$$\bar{g}^{ij}_{\alpha\beta\bar{|}k} = e^{-2\sigma}\left\{g^{ij}_{\alpha\beta|k} - \frac{\partial g^{ij}_{\alpha\beta}}{\partial \dot{x}^l_\alpha} U^l_{pk}\, \dot{x}^p_\alpha + g^{ij}_{\alpha\beta}\, U^i_{kl} + g^{il}_{\alpha\beta}\, U^j_{kl}\right\} + \frac{\partial e^{-2\sigma}}{\partial x^k}\, g^{ij}_{\alpha\beta} \quad \ldots(3.7)$$

and

$$\bar{C}^{ijk}_{\alpha\beta\gamma\bar{|}m} = e^{-2\sigma}\left\{C^{ijk}_{\alpha\beta\gamma|m} - \frac{\partial C^{ijk}_{\alpha\beta\gamma}}{\partial \dot{x}^l_\alpha} U^l_{pm}\, \dot{x}^p_\alpha + C^{ljk}_{\alpha\beta\gamma}\, U^i_{ml} + C^{ilk}_{\alpha\beta\gamma}\, U^j_{ml} + C^{ijl}_{\alpha\beta\gamma}\, U^k_{ml}\right\} + \frac{\partial e^{-2\sigma}}{\partial x^m}\, C^{ijk}_{\alpha\beta\gamma}. \quad \ldots(3.8)$$

PROOF : In view of (1.6) the covariant derivative of $g^{ij}_{\alpha\beta}$ is given by

$$g^{ij}_{\alpha\beta|k} = \frac{\partial g^{ij}_{\alpha\beta}}{\partial x^k} - \frac{\partial g^{ij}_{\alpha\beta}}{\partial \dot{x}^l_\alpha} \Gamma^l_{pk}\, \dot{x}^p_\alpha + \Gamma^i_{kl}\, g^{lj}_{\alpha\beta} + \Gamma^j_{kl}\, g^{il}_{\alpha\beta}. \quad \ldots(3.9)$$

Similarly in $\bar{A}^{(m)}_n$, we have

$$\bar{g}^{ij}_{\alpha\beta\bar{|}k} = \frac{\partial \bar{g}^{ij}_{\alpha\beta}}{\partial x^k} - \frac{\partial \bar{g}^{ij}_{\alpha\beta}}{\partial \dot{x}^l_\alpha} \bar{\Gamma}^l_{pk}\, \dot{x}^p_\alpha + \bar{\Gamma}^i_{kl}\, \bar{g}^{lj}_{\alpha\beta} + \bar{\Gamma}^j_{kl}\, \bar{g}^{il}_{\alpha\beta}. \quad \ldots(3.10)$$

Using (2.2) and (2.6) in the right-hand side of (3.10) and then simplifying the result in the light of (3.9), we get the relation (3.7).

Proceeding on the same lines as for the first part (3.7), we get (3.8).

Corollary 3.3*a* — Under homothetic transformation the tensors $g^{ij}_{\alpha\beta|k}$ and $\bar{g}^{ij}_{\alpha\beta\bar{|}k}$ are proportional.

Corollary 3.3*b* — Under homothetic transformation the tensors $C^{ijk}_{\alpha\beta\gamma|m}$ and $\bar{C}^{ijk}_{\alpha\beta\gamma\bar{|}m}$ are proportional.

Theorem 3.4 — If $A^{(m)}_n$ and $\bar{A}^{(m)}_n$ are in conformal correspondence, we have

$$\bar{g}^{ij}_{\alpha\beta\,\bar{\|}\,k} = e^{-2\sigma}\left\{ g^{ij}_{\alpha\beta\,\|\,k} - \frac{\partial g^{ij}_{\alpha\beta}}{\partial \dot{x}^{l}_{\alpha}} U^{l}_{kp}\, \dot{x}^{p}_{\alpha} + g^{lj}_{\alpha\beta}\, U^{i}_{lk} + g^{il}_{\alpha\beta}\, U^{j}_{lk} \right\} + \frac{\partial e^{-2\sigma}}{\partial x^{k}}\, g^{ij}_{\alpha\beta} \qquad \ldots(3.11)$$

and

$$\bar{C}^{ijk}_{\alpha\beta\gamma\,\bar{\|}\,m} = e^{-2\sigma}\left\{ C^{ijk}_{\alpha\beta\gamma\,\|\,m} - \frac{\partial C^{ijk}_{\alpha\beta\gamma}}{\partial \dot{x}^{l}_{\alpha}} U^{l}_{mp}\, \dot{x}^{p}_{\alpha} + C^{ljk}_{\alpha\beta\gamma}\, U^{i}_{lm} + C^{ilk}_{\alpha\beta\gamma}\, U^{j}_{lm} + C^{ijl}_{\alpha\beta\gamma}\, U^{k}_{lm} \right\} + \frac{\partial e^{-2\sigma}}{\partial x^{m}}\, C^{ijk}_{\alpha\beta\gamma}. \qquad \ldots(3.12)$$

PROOF : In view of (1.7), the covariant derivative of $g^{ij}_{\alpha\beta}$ is given by

$$g^{ij}_{\alpha\beta\,\|\,k} = \frac{\partial g^{ij}_{\alpha\beta}}{\partial x^{k}} - \frac{\partial g^{ij}_{\alpha\beta}}{\partial \dot{x}^{l}_{\alpha}} \Gamma^{l}_{kp}\, \dot{x}^{p}_{\alpha} + \Gamma^{i}_{lk}\, g^{lj}_{\alpha\beta} + \Gamma^{j}_{lk}\, g^{il}_{\alpha\beta}\,. \qquad \ldots(3.13)$$

Similarly in $\bar{A}^{(m)}_{n}$, we have

$$\bar{g}^{ij}_{\alpha\beta\,\bar{\|}\,k} = \frac{\partial \bar{g}^{ij}_{\alpha\beta}}{\partial x^{k}} - \frac{\partial \bar{g}^{ij}_{\alpha\beta}}{\partial \dot{x}^{l}_{\alpha}} \bar{\Gamma}^{l}_{kp}\, \dot{x}^{p}_{\alpha} + \bar{\Gamma}^{i}_{lk}\, \bar{g}^{lj}_{\alpha\beta} + \bar{\Gamma}^{j}_{lk}\, \bar{g}^{il}_{\alpha\beta}\,. \qquad \ldots(3.14)$$

Using (2.2) and (2.6*a*) in the right-hand side of (3.14) and then simplifying the result in view of (3.13), we get the relation (3.11). Proceeding on the same lines as above, we get (3.12).

Corollary 3.4*a* — Under homothetic transformation the tensors $g^{ij}_{\alpha\beta\,\|\,k}$ and $\bar{g}^{ij}_{\alpha\beta\,\bar{\|}\,k}$ are proportional.

Corollary 3.4*b* — Under homothetic transformation the tensors $C^{ijk}_{\alpha\beta\gamma\,\|\,m}$ and $\bar{C}^{ijk}_{\alpha\beta\gamma\,\bar{\|}\,m}$ are proportional.

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