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ON ĆIRIĆ TYPE MAPS WITH A NONUNIQUE FIXED POINT

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In this paper nonunique fixed point theorems have been established for Ćirić type maps on a orbitally complete metric space into itself.

1. INTRODUCTION

Recently, Ćirić (1974) proved some nonunique fixed point theorems for orbitally continuous self mappings T on M which satisfy a condition of the type

$$\min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(y, Tx)\} \leqslant qd(x, y),$$

for all $x, y \in M$ and for some $q \in (0, 1)$. The purpose of the present paper is to establish some fixed point theorems for mappings T on M which are not necessarily continuous and which satisfy a condition of the type

$$\min\{[d(Tx, Ty)]^2, d(x, y)\, d(Tx, Ty), [d(y, Ty)]^2\} - \min\{d(x, Tx)\, d(y, Ty), d(x, Ty)\, d(y, Tx)\} \leqslant qd(x, Tx)\, d(y, Ty),$$

for all $x, y \in M$ and for some $q \in (0, 1)$.

2. MAIN RESULTS

Let (M, d) be a metric space. We recall that (see Ćirić 1974) a mapping T on M is orbitally continuous if $\lim_i T^{n_i} x = u$ implies $\lim_i TT^{n_i} x = Tu$ for each $x \in M$. A space M is T-orbitally complete if every Cauchy sequence of the form $\{T^{n_i} x\}_{i=1}^{\infty}$, $x \in M$, converges in M. Our main result in this paper reads as follows.

Theorem 1 — Let $T : M \to M$ be an orbitally continuous mapping on M and let M be T-orbitally complete. If T satisfies the following condition

$$\min\{[d(Tx, Ty)]^2, d(x, y)\, d(Tx, Ty), [d(y, Ty)]^2\} - \min\{d(x, Tx)\, d(y, Ty), d(x, Ty)\, d(y, Tx)\} \leqslant qd(x, Tx)\, d(y, Ty) \quad \ldots(1)$$

for all $x, y \in M$ and $q \in (0, 1)$, then for each $x \in M$, the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to a fixed point of T.

PROOF : Let $x \in M$ be arbitrary. We define a sequence

$$x_0 = x,\ x_1 = Tx_0,\ x_2 = Tx_1, \ldots, x_n = Tx_{n-1}, \ldots . \qquad \ldots(2)$$

If for some n, $x_n = x_{n+1}$, then $\{x_n\}$ is a Cauchy sequence, and the limit of $\{x_n\}$ is a fixed point of T. Suppose that $x_n \neq x_{n+1}$ for each $n = 0, 1, 2, \ldots$. By (1) for $x = x_{n-1}$ and $y = x_n$ we have

$$\min\{[d(x_n, x_{n+1})]^2, d(x_{n-1}, x_n)\, d(x_n, x_{n+1}), [d(x_n, x_{n+1})]^2\} - \min\{d(x_{n-1}, x_n)\, d(x_n, x_{n+1}), 0\} \leqslant qd(x_{n-1}, x_n)\, d(x_n, x_{n+1})$$

i.e.

$$\min\{[d(x_n, x_{n+1})]^2, d(x_{n-1}, x_n)\, d(x_n, x_{n+1})\} \leqslant qd(x_{n-1}, x_n)\, d(x_n, x_{n+1}).$$

Since

$$d(x_{n-1}, x_n)\, d(x_n, x_{n+1}) \leqslant qd(x_{n-1}, x_n)\, d(x_n, x_{n+1})$$

is impossible (as $q < 1$), one has

$$d(x_n, x_{n+1}) \leqslant qd(x_{n-1}, x_n).$$

Proceeding in this manner we obtain

$$d(x_n, x_{n+1}) \leqslant qd(x_{n-1}, x_n) \leqslant q^2 d(x_{n-2}, x_{n-1}) \leqslant \ldots \leqslant q^n d(x, Tx).$$

Hence for any $p \in I^+$ one has

$$d(x_n, x_{n+p}) \leqslant \sum_{k=n}^{n+p-1} d(x_k, x_{k+1}) \leqslant \left(\sum_{k=n}^{n+p-1} q^k \right) d(x, Tx) \leqslant \frac{q^n}{1-q}\, d(x, Tx).$$

Since $\lim_n q^n = 0$ it follows that (2) is the Cauchy's sequence. M being T-orbitally complete, there is some $u \in M$ such that $u = \lim_n T^n x$. By orbital continuity of T

$$Tu = \lim_n TT^n x = u,$$

i.e. u is a fixed point of T. The proof of the theorem is complete.

Recently, Achari (1976) has obtained a localized version of Ćirić's fixed point theorem (Ćirić 1974, Theorem 1). Our next result deals with a localized version of our Theorem 1.

Theorem 2 — Let

$$B = B(x_0, r)\, \{x \in M \mid d(x_0, x) \leqslant r\}$$

where (M, d) is a orbitally complete metric space. Let T be an orbitally continuous mapping of B into M and satisfies (1) for $x, y \in B$ and

$$d(x_0, Tx_0) \leqslant (1 - q)\, r. \qquad \ldots(3)$$

Then T has a fixed point.

PROOF: By (3) we have

$$x_1 = Tx_0 \in B(x_0, r)$$

and by (1) for $x = x_0$ and $y = x_1$, we have

$$\min\{[d(x_1, x_2)]^2, d(x_0, x_1)\, d(x_1, x_2)\, [d(x_1, x_2)]^2\}$$
$$- \min\{d(x_0, x_1)\, d(x_1, x_2), d(x_0, x_2)\, d(x_1, x_1)\} \leqslant q d(x_0, x_1)\, d(x_1, x_2)$$

which implies

$$d(x_1, x_2) \leqslant q d(x_0, x_1) \leqslant q(1 - q)\, r.$$

Hence

$$d(x_0, x_2) \leqslant d(x_0, x_1) + d(x_1, x_2), \ (x_2 = Tx_1)$$
$$\leqslant (1 - q)\, r + q(1 - q)\, r = (1 + q)\,(1 - q)\, r.$$

Suppose that

$$d(x_0, x_n) \leqslant (1 + q + \dots + q^{n-1})\,(1 - q)\, r$$

and that

$$d(x_{n-1}, x_n) \leqslant q^{n-1}(1 - q)\, r, \ (x_n = Tx_{n-1}).$$

Then by (1) for $x = x_{n-1}$ and $y = x_n$, we have

$$\min\{[d(x_n, x_{n+1})]^2, d(x_{n-1}, x_n)\, d(x_n, x_{n+1}), [d(x_n, x_{n+1})]^2\}$$
$$- \min\{d(x_{n-1}, x_n)\, d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1})\, d(x_n, x_n)\}$$
$$\leqslant q d(x_{n-1}, x_n)\, d(x_n, x_{n+1})$$

which implies

$$d(x_n, x_{n+1}) \leqslant q d(x_{n-1}, x_n) \leqslant q^n(1 - q)\, r.$$

Therefore

$$d(x_0, x_{n+1}) \leqslant d(x_0, x_n) + d(x_n, x_{n+1})$$
$$\leqslant (1 + q + \dots + q^{n-1})\,(1 - q)\, r + q^n(1 - q)\, r$$
$$= (1 + q + \dots + q^n)\,(1 - q)\, r \leqslant r.$$

Thus the sequence x_0, $x_{n+1} = Tx_n$, $n \geqslant 0$ is contained in B. Also

$$d(x_n, x_m) \leqslant d(x_n, x_{n+1}) + \dots + d(x_{m-1}, x_m)$$
$$\leqslant (q^n + \dots + q^{m-1})\,(1 - q)\, r \leqslant q^n r \to 0.$$

Since B is also orbitally complete, so $u = \lim_n T^n x$ for some $u \in B$. By orbital continuity of T we have

$$Tu = \lim_n TT^n x = u.$$

Thus u is a fixed point of T. This completes the proof of the theorem.

To this end we establish the following theorem initiated by Maia's (1968) fixed point theorem.

Theorem 3 — Let M be a metric space with two metrices d and δ. If M satisfies the following conditions :

(i) $d(x, y) \leqslant \delta(x, y)$ for every x, y in M,

(ii) M is orbitally complete with respect to d,

(iii) the mapping $T : M \to M$ is orbitally continuous with respect to d, and

$$\min \{[\delta(Tx, Ty)]^2, \delta(x, y)\, \delta(Tx, Ty), [\delta(y, Ty)]^2\} - \min \{\delta(x, Tx)\, \delta(y, Ty), \delta(x, Ty)\, \delta(y, Tx)\} \leqslant q\delta(x, Tx)\, \delta(y, Ty), \quad \ldots(4)$$

for all $x, y \in M$ and $q \in (0, 1)$, then T has a fixed point in X.

PROOF : Let $x_0 \in X$ be arbitrary and define a sequence $\{x_n\}$ as in (2). By (4) for $x = x_{n-1}$ and $y = x_n$, we have

$$\min \{[\delta(x_n, x_{n+1})]^2, \delta(x_{n-1}, x_n)\, \delta(x_n, x_{n+1}), [\delta(x_n, x_{n+1})]^2\} - \min \{\delta(x_{n-1}, x_n)\, \delta(x_n, x_{n+1}), \delta(x_{n-1}, x_{n+1})\, \delta(x_n, x_n)\} \leqslant qd(x_{n-1}, x_n)\, \delta(x_n, x_{n+1})$$

which implies

$$\delta(x_n, x_{n+1}) \leqslant q\delta(x_{n-1}, x_n).$$

Proceeding in this way

$$\delta(x_n, x_{n+1}) \leqslant q\delta(x_{n-1}, x_n) \leqslant \ldots \leqslant q^n\delta(x_0, x_1)$$

and hence

$$\delta(x_n, x_{n+p}) \leqslant \frac{q^n}{1-q}\, \delta(x_0, x_1)$$

where p is any positive integer. Therefore, by $d \leqslant \delta$, we have

$$d(x_n, x_{n+p}) \leqslant \frac{q^n}{1-q}\, \delta(x_0, x_1).$$

This shows that the sequence $\{x_n\}$ is a Cauchy sequence with respect to d. M being T-orbitally complete, there is some $u \in M$ such that $u = \lim_n T^n x$. By orbital continuity of T

$$Tu = \lim_{n} TT^n x = u.$$

Thus u is a fixed point of T and hence the proof of the theorem is complete.

Finally we note that the conclusions of Theorems 1 and 2 remains valid if we replace the condition (1) by

$$\min \{[d(Tx, Ty)]^2, d(x, y)\, d(Tx, Ty), [d(y, Ty)]^2\}$$
$$- \min \{d(x, Ty), d(y, Tx)\} \leqslant q d(x, Tx)\, d(y, Ty) \qquad \ldots(5)$$

for all $x, y \in M$ and $q \in (0, 1)$. We also note that the conclusion of Theorem 3 remains valid by using the modified version of condition (5).

References

Achari, J. (1976). On Ćirić's nonunique fixed points. *Math. Vesnik*, **13**, 255-57.

Ćirić, Lj. B. (1974). On some maps with a nonunique fixed point. *Pub. Inst. Math.*, **17**, 52-58.

Maia, M. G. (1968). Un'osservazione sulle contrazioni metriche. *Rend. Sem. Mat. Padova*, **40**, 139-43.