Indian J. pure appl. Math., **10**(8) : 1044-1049, August 1979

FORMULAS INVOLVING PRODUCTS OF CLASSICAL ORTHOGONAL POLYNOMIALS BY FUJIWARA-THAKARE APPROACH

B. K. KARANDE

Department of Mathematics, Science College, Karad, Dist. Satara

(*Received* 4 *December* 1978)

In this paper, formulas involving products of extended Jacobi polynomials are obtained. In the limiting cases, formulas involving products of classical orthogonal polynomials are deduced. Results for Bessel polynomials seem to be new one.

1. INTRODUCTION

Let $F_n(\alpha, \beta, x)$ be the sequence of extended Jacobi polynomials defined by the Rodrigues' Formula

$$F_n(\alpha, \beta, x) = \frac{1}{K_n W(x)} D^n\{W(x)\, X^n(x)\},\ D \equiv \frac{d}{dx} \quad ...(1)$$

with

$$W(x) = \frac{(x-a)^\alpha (b-x)^\beta}{(b-a)^{\alpha+\beta+1} B(\alpha+1, \beta+1)} \ (\alpha > -1, \beta > -1)$$

$$X(x) = c(x-a)(b-x),\ (c > 0), \quad \text{and} \quad k_n = (-1)^n n!$$

These polynomials $\{F_n(\alpha, \beta, x)\}$ are orthogonal with respect to the above weight function $W(x)$ over the interval (a, b), a and b are real numbers with $a < b$.

In what follows the parameter λ is involved and is connected by the relation $\lambda = c(b-a)$. The Fujiwara-Thakare approach essentially gives the following interesting relationships between the classical orthogonal polynomials and the extended Jacobi polynomials (see Fujiwara 1967, Thakare 1972, 1978).

(A) *Jacobi polynomials* : When $-a = b = \lambda = 1$, we have

$$F_n(\alpha, \beta, x) = P_n^{(\beta,\alpha)}(x). \quad ...(2)$$

(B) *Lagueree polynomials* : When $a = 0$, $\beta = b$ and $\lambda = 1$, we have

$$\lim_{b\to\infty} F_n(\alpha, b, x) = (-1)^n L_n^\alpha(x). \quad ...(3)$$

(C) *Hermite polynomials*: When $\beta = \alpha$, $-a = b = \sqrt{\alpha}$ ($\alpha > 0$), and in view of $\lambda \to 2/\sqrt{\alpha}$ (see Fujiwara 1967, and Thakare 1972, 1978), we obtain

$$\lim_{\alpha \to \infty} F_n(\alpha, \alpha, x) = \frac{H_n(x)}{n!}. \qquad \ldots(4)$$

In fact $F_n(\alpha, \beta, x)$ also yield as a particular case the Bessel polynomials $Y_n(x; r, s)$ in the following manner.

(D) *Bessel polynomials*: When $-a = b = \lambda = 1$, we have

$$\lim_{\epsilon \to \infty} \frac{\Gamma(n+1)}{\epsilon^n} F_n\left(r - \epsilon - 1, \epsilon - 1; 1 + \frac{2x\epsilon}{s}\right) = Y_n(x; r, s) \qquad \ldots(5)$$

The main purpose of this note is to obtain formulas involving products of Jacobi polynomials, Laguerre polynomials, Hermite polynomials and Bessel polynomials by Fujiwara-Thakare approach. Incidentally, this approach leads to new results for Bessel polynomials.

2. Products of $F_n(\alpha, \beta, x)$

Let us recall that formula (1) yields

$$F_n(\alpha, \beta, x) = c^n(1+\alpha)_n \sum_{k=0}^{n} \frac{(-1)^n \, \Gamma(1+\beta+n)\,(x-a)^k\,(x-b)^{n-k}}{k!\,(n-k)!\,(1+\alpha)_k\,\Gamma(1+\beta+n-k)}.$$

Using this, and after a straight forward computation one arrives at the following generating relation.

$$\sum_{n=0}^{\infty} \frac{F_n(\alpha, \beta - n; x)\, t^n}{c^n(1+\alpha)_n} = \sum_{n,k=0}^{\infty} \frac{(-1)^n\,(-\beta)_k\,(x-a)^k\,(x-b)^k\,t^{n+k}}{k!\,n!\,(1+\alpha)_k}$$

$$= e^{(b-x)t}\,{}_1F_1\,[-\beta; 1+\alpha; (x-a)\,t]. \qquad \ldots(6)$$

Bailey (1939) has stated the following formula:

$${}_1F_1\,[a; c; u+v] = \sum_{r=0}^{\infty} \frac{(a)_r\,(c-a)_r\,(uv)^r}{r!(c+r-1)_r\,(c)_{2r}}$$

$$\times\ {}_1F_1\,[a+r; c+2r; u]\,{}_1F_1\,[a+r; c+2r; v]. \qquad \ldots(7)$$

From (6) and (7) we obtain

$$\sum_{k=0}^{\infty} \frac{(u+v)^k F_n(\alpha, \beta-k, x)}{c^k(1+\alpha)_k} t^k$$

$$= \exp[(b-x)(u+v)] \sum_{r=0}^{\infty} \frac{(-\beta)_r (1+\alpha+\beta)_r (x-a)^{2r} (uv)^r}{r!\,(\alpha+r)_r (1+\alpha)_{2r}}$$

$$\times {}_1F_1[-\beta+r; 1+\alpha+2r; u(x-a)]\, {}_1F_1[-\beta+r; 1+\alpha+2r; v(x-a)]$$

$$= \sum_{r=0}^{\infty} \frac{(-\beta)_r (1+\alpha+\beta)_r (x-a)^{2r} (uv)^r}{r!\,(\alpha+r)_{2r} (1+\alpha)_r} \cdot \sum_{m=0}^{\infty} \frac{u^m F_m(\alpha+2r, \beta-r-m, x)}{c^m(\alpha+2r+1)_m}$$

$$\times \sum_{n=0}^{\infty} \frac{v^n F_n(\alpha+2r; \beta-r-n; x)}{c^n(\alpha+2r+1)_n}$$

$$= \sum_{m,n=0}^{\infty} \sum_{r=0}^{\min(m,n)} \frac{(-\beta)_r (1+\alpha+\beta)_r (x-a)^{2r} (uv)^r u^m v^n}{r!\,(\alpha+r)_{2r} (1+\alpha)_{1r}\, c^{m+n-2r}}.$$

$$\times \frac{F_{m-r}(\alpha+2r, \beta-m; x)\, F_{n-r}(\alpha+2r, \beta-n; x)}{(1+\alpha+2r)_{m-r} (1+\alpha+2r)_{n-r}}.$$

Thus one obtains

$$\binom{m+n}{n} F_{m+n}(\alpha, \beta-m-n, x)$$

$$= \frac{(1+\alpha)_{m+n}}{(1+\alpha)_n (1+\alpha)_m} \sum_{r=0}^{\min(m,n)} \frac{(-1)^r \binom{\beta}{r} (1+\alpha)_{2r}}{(\alpha+r)_r (1+\alpha+m)_r}$$

$$\times \frac{(1+\alpha+\beta)_r}{(1+\alpha+n)_r} (x-a)^{2r} c^{2r} F_{m-r}(\alpha+2r, \beta-m, x)$$

$$\times F_{n-r}(\alpha+2r, \beta-n, x). \qquad \ldots(8)$$

Such results were also obtained earlier by Patil and Thakare (1977), and the author (Karande 1975) by different methods.

Next we recall the following formula of Bailey (1936):

$${}_1F_1[a; c; u]\, {}_1F_1[a; c; v] = \sum_{r=0}^{\infty} \frac{(-1)^r (a)_r (c-a)_r (uv)^r}{r!\,(c)_r (c)_{2r}}$$

$$\times {}_1F_1(a+r; c+2r; u+v).$$

As a consequence of this result and result (6), we obtain

$$e^{-(b-x)(u+v)} \sum_{m,n=0}^{\infty} \frac{F_n(\alpha, \beta - n; x) F_m(\alpha, \beta - m; x)(u+v)^{m+n}}{c^{m+n}(1+\alpha)_n (1+\alpha)_m}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (-\beta)_r (1+\alpha+\beta)_r (x-a)^{2r} (uv)^r \exp[-(b-x)(u+v)]}{r!\,(1+\alpha)_r (1+\alpha)_{2r}}.$$

$$\times \sum_{m=0}^{\infty} \frac{F_m(\alpha+2r, \beta-m-r; x)(u+v)^m}{c^m (1+\alpha+2r)_m}.$$

Thus, we get

$$F_m(\alpha; \beta - m; x) F_n(\alpha, \beta - n; x) = \frac{(1+\alpha)_n (1+\alpha)_m}{(1+\alpha)_{m+n}}$$

$$\times \sum_{r=0}^{\min(m,n)} \binom{\beta}{r} \binom{m+n-2r}{m-r} \frac{(1+\alpha+\beta)_r}{(1+\alpha)_r} [c(x-a)]^{2r}$$

$$\times F_{m+n-2r}(\alpha+2r, \beta-m-n+r; x). \qquad \ldots(9)$$

3. Particular Cases

As a consequence of Fujiwara-Thakare relationships through (2) to (5), one obtains from (8) the following known results (Carlitz 1956, Chatterjea 1968) (the result for Bessel polynomials seems to be new):

(A) *Jacobi polynomials*

$$\binom{m+n}{n} P_{m+n}^{(\alpha,\beta-m-n)} = \frac{(1+\alpha)_{m+n}}{(1+\alpha)_m (1+\alpha)_n}$$

$$\times \sum_{r=0}^{\min(m,n)} (-1)^r \binom{\beta}{r} \frac{(1+\alpha)_r (1+\alpha+\beta)_r}{(\alpha+r)_r}$$

$$\times \frac{1}{(1+\alpha+m)_r (1+\alpha+n)_r} \left(\frac{1-x}{2}\right)^{2r} P_{m-r}^{(\alpha+2r,\beta-m)}(x)$$

$$\times P_{n-r}^{(\alpha+2r,\beta-n)}(x).$$

(B) *Lagueree polynomials*

$$\binom{m+n}{n} L_{m+n}^{\alpha}(x) = \frac{(1+\alpha)_{m+n}}{(1+\alpha)_n (1+\alpha)_m} \times$$

(*equation continued on p.* 1048)

$$\times \sum_{r=0}^{\min(m,n)} \frac{(-1)^r (1+\alpha)_{2r} x^{2r}}{r!\,(\alpha+r)_r\,(\alpha+m+1)_r\,(\alpha+n+1)_r}$$

$$\times L_{m-r}^{\alpha+2r}(x)\, L_{n-r}^{\alpha+2r}(x).$$

(C) *Hermite polynomials*

$$H_{m+n}(x) = \sum_{r=0}^{\min(m,n)} (-2)^r \binom{m}{r}\binom{n}{r} r!\, H_{m-r}(x)\, H_{n-r}(x).$$

(D) *Bessel polynomials*

$$Y_{m+n}(x;\, r,\, s) = \sum_{p=0}^{\min(m,n)} \binom{n}{p}\binom{m}{p} p!\,(r-1)_p \left(\frac{x}{s}\right)^{2p}$$

$$\times Y_{m-p}(x;\, r+2p,\, s)\, Y_{n-p}(x;\, r+2p,\, s).$$

In the same fashion one also obtains the following particular cases from formula (9). The result involving Bessel polynomials seems to be new.

(A) *Jacobi polynomials*

$$P_n^{(\alpha,\beta-n)}(x)\, P_m^{(\alpha,\beta-m)}(x) = \frac{(1+\alpha)_n\,(1+\alpha)_m}{(1+\alpha)_{m+n}} \sum_{r=0}^{\min(m,n)} \binom{\beta}{r}\binom{m+n-2r}{m-r}$$

$$\times \frac{(1+\alpha+\beta)_r}{(1+\alpha)_r} \left(\frac{1-x}{2}\right)^{2r} P_{m+n-2r}^{(\alpha+2r,\beta-m-n+r)}(x).$$

(B) *Laguerre polynomials*

$$L_m^{\alpha}(x)\, L_n^{\alpha}(x) = \frac{\Gamma(1+\alpha+m)\,\Gamma(1+\alpha+n)}{\Gamma(1+\alpha+m+n)}$$

$$\times \sum_{r=0}^{\min(m,n)} \frac{(m+n-2r)!\, x^{2r} L_{m+n-2r}^{\alpha+2r}(x)}{r!\,(m-r)!\,\Gamma(1+\alpha+r)}.$$

(C) *Hermite polynomials*

$$H_m(x)\, H_n(x) = \sum_{r=0}^{\min(m,n)} 2^r \binom{m}{r}\binom{n}{r} r!\, H_{m+n-2r}(x).$$

(D) *Bessel polynomials*

$$Y_n(x; r, s)\, Y_m(x; r, s) = \sum_{p=0}^{\min(m,n)} (-1)^p \binom{n}{p} \binom{m}{p} p!\, (r-1)_p \left(\frac{x}{s}\right)^{2p} \times Y_{m+n-2p}(x; r+2p, s).$$

References

Balley, W. N. (1936). On product of two Legendre polynomials with different arguments. *Proc. Lond. math. Soc.*, **41**, 215-20.

———— (1939). On product of two Lagueree polynomials. *Q. Jl Math.*, **10**, 60-66.

Carlitz, L. (1956). On Jacobi polynomials. *Boll. Unione Matematica Italinna, Series III*, **11**, 371-81.

Chatterjea S. K. (1968). Operational results connected with some classical polynomials. *Acta. Math. Acad. Sci. Hung.*, **19**, 53-58.

Fujiwara, Izuru (1967). A unified presentation of classical orthogonal polynomials. *Math. Japan*, **11**, 133-48.

Karande, B. K. (1975). A study of certain sets of polynomials. Ph.D. thesis, Shivaji University, Kolhapur.

Patil, K. R., and Thakare, N. K. (1977). Operational formulas and generating functions in the unified form for the classical orthogonal polynomials. *Math. Student*, **45**, 41-51.

Thakare, N. K. (1972). A study of certain sets of orthogonal polynomials and their applications, Ph.D. thesis, Shivaji University, Kolhapur.

———— (1978). A Unified Presentation of the Study of Classical Orthogonal Polynomials. (Monograph, under publication).