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ON THE CONVERGENCE OF SEQUENCES OF FIXED POINTS IN 2-METRIC SPACES

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Theorems on the convergence of sequences of fixed points in 2-metric spaces, related to the work of Rhoades (1978) are presented. These are in fact, generalizations of several previously known results.

1. Introduction

The notion of 2-metric space was introduced by Gähler (1963/64). Further development of this branch of mathematics, in recent years, is due to Diminie, White, Ehret, Iśeki, Siddiqi, Gupta and Gähler himself.

More recently Iśeki (1975a, b), Iśeki *et al.* (1976) and Rhoades (1978) studied the aspects of fixed point theory in 2-metric spaces. In this note some results on fixed point theorems of mappings considered by Rhoades have been obtained.

Following Gähler (1963/64) and White (1969) we have following definition:

Definition 1 — A 2-mertic space is a space X in which for each triple of points a, b, c there exists a real-valued function $\rho(a, b, c)$ such that following hold:

(i) To each pair of points a, b with $a \neq b$ in X there exists a point $c \in X$ such that $\rho(a, b, c) \neq 0$;

(ii) $\rho(a, b, c) = 0$ when at least two of the points are equal;

(iii) $\rho(a, b, c) = \rho(b, c, a) = \rho(a, c, b)$;

(iv) $\rho(a, b, c) \leqslant \rho(a, b, d) + \rho(a, d, c) + \rho(d, b, c)$.

It is easily seen that ρ is non-negative.

Definition 2 — A sequence $\{x_n\}$ in X is 'convergent' and $x \in X$ is the limit of the sequence if $\lim_{n \to \infty} \rho(x_n, x, a) = 0$ for all $a \in X$.

Definition 3 — A sequence $\{x_n\}$ in a 2-metric space X is called 'Cauchy sequence' if $\lim_{n, m \to \infty} \rho(x_n, x_m, a) = 0$ for all $a \in X$.

If every Cauchy sequence is convergent, X is called a 'complete 2-metric space'.

Definition 4 — A 2-metric which is continuous in all its three arguments is called 'continuous'.

The following results are due to Rhoades (1978) :

Theorem A — Let X be a complete 2-metric space, $T : X \to X$ satisfying : "there exists a q, $0 \leqslant q < 1$ such that for each $x, y, a \in X$

$$(*)\ \rho(Tx, Ty, a) \leqslant q \max \{\rho(x, y, a), \rho(x, Tx, a), \rho(y, Ty, a),$$
$$\rho(x, Ty, a), \rho(y, Tx, a)\}",$$

then T possesses a unique fixed point z and $\lim_n T^n x_0 = z$ for each $x_0 \in X$.

Theorem B — Let X be a complete 2-metric space with ρ continuous, $\{T_n\}$ as sequence of mapping $T_n : X \to X$ satisfying (*) for each n and same q, such that T_n tends pointwise to a function T. Then T has a unique fixed point z and $z_n \to z$, where the z_n are the unique fixed points of T_n.

It may be remarked that Theorem A and Theorem B are 2-metric space versions of fixed point theorems due to Čirič (1971, 1974). Also the contractive definition (*) is one of the most general definitions possible for 2-metric spaces.

2. Main Results

Theorem 1 — Let (X, ρ_0) be a 2-metric space. ρ_n is sequence of 2-metrics on X converging uniformly to ρ_0. Let $\{T_n\}$ be a sequence of self-mappings on X converging ρ_0-pointwise to a map T with fixed point z and let T_n having fixed points z_n satisfying (*) with respect to ρ_n for each n and same q. Then $z_n \to z$.

Proof : The uniform convergence of ρ_n to ρ_0 implies that for any $\epsilon > 0$ and $x, y, a \in z$ one gets

$$| \rho_n(x, y, a) - \rho_0(x, y, a) | < \left(\frac{1-q}{3+q}\right)\epsilon.$$

Also by the pointwise convergence of T_n to T with respect to the 2-metric ρ_0 yields that for all $a \in X$

$$\rho_0(Tz, T_n z, a) < \left(\frac{1-q}{3+q}\right)\epsilon$$

whenever $n \geqslant N$ for some natural number N. Now for $n \geqslant N$, we have

$$\rho_0(z, z_n, a) = \rho_0(Tz, T_n z_n, a)$$
$$\leqslant \rho_0(Tz, T_n z_n, T_n z) + \rho_0(Tz, T_n z, a) + \rho_0(T_n z, T_n z_n, a)$$

(*equation continued on p.* 1064)

$$\leqslant \rho_n(Tz, T_n z_n, T_n z) + \rho_0(Tz, T_n z, a)$$
$$+ \rho_n(T_n z, T_n z_n, a) + \left(\frac{1-q}{3+q}\right)\epsilon + \left(\frac{1-q}{3+q}\right)\epsilon.$$

From the proof of Theorem 2 of Rhoades (1978), we conclude

(A) $$\rho_n(T_n z, T_n z_n, a) \leqslant q \max\{\rho_n(z, z_n, a), \rho_n(z, T_n z, a)\}.$$

Hence

$$\rho_n(Tz, T_n z_n, T_n z) = \rho_n(T_n z_n, T_n z, Tz) = 0.$$

Considering two cases of (A), we get

$$\rho_n(T_n z, T_n z_n, a) \leqslant q\rho_n(z, z_n, a)$$
$$\leqslant q\left[\rho_0(z, z_n, a) + \left(\frac{1-q}{3+q}\right)\epsilon\right].$$

Here second alternative in (A) is not admissible.

Thus

$$\rho_0(z, z_n, a) \leqslant \rho_0(Tz, T_n z, a) + \left[q\rho_0(z, z_n, a) + q\left(\frac{1-q}{3+q}\right)\epsilon\right]$$
$$+ \left(\frac{1-q}{3+q}\right)\epsilon + \left(\frac{1-q}{3+q}\right)\epsilon$$

$$\rho_0(z, z_n, a) \leqslant \frac{\rho_0(Tz, T_n z, a)}{(1-q)} + \left(\frac{2+q}{1-q}\right)\left(\frac{1-q}{3+q}\right)\epsilon < \epsilon$$

which implies the convergence of z_n to z.

Remarks : (1) Theorem B is a particular case of our Theorem 1 when $\rho_n = \rho_0$ for each n.

(2) The existence of the unique fixed point z of T can be proved under the extra assumption that ρ_n is continuous for each n and via the use of condition (*).

Theorem 2 — Let $\{T_n\}$ and $\{S_n\}$ be sequences of mappings of a 2-metric space X with ρ continuous and for each n satisfying

$$\rho(T_n x, S_n y, a) \leqslant q \max\{\rho(x, T_n x, a), \rho(y, S_n y, a), \rho(x, y, a),$$
$$\rho(x, S_n y, a), \rho(y, T_n x, a)\}$$

for all $x, y, a \in X$, q a fixed constant satisfying $0 \leqslant q < 1$. If $\{T_n\}$ and $\{S_n\}$ converge pointwise to mappings T and S on X respectively, the following three statements are equivalent :

(i) T has a fixed point,

(ii) S has a fixed point,

(iii) $\{x_n\}$ is a convergent sequence where $\{x_n\}$ is a common fixed point of T_n and S_n for each n.

PROOF : We shall prove the theorem by the use of following lemmas.

Lemma 1 — If x_n and x are fixed points of S_n and T respectively, and if $\{T_n\}$ converges pointwise to T, then $\{x_n\}$ converges to x.

PROOF : For each natural number n, consider

$$\rho(x, x_n, a) = \rho(Tx, S_n x_n, a)$$
$$\leqslant \rho(Tx, S_n x_n, T_n x) + \rho(Tx, T_n x, a) + \rho(T_n x, S_n x_n, a).$$

From the given condition

$$\rho(T_n x, S_n x_n, a) \leqslant q \max \{\rho(x, x_n, a), \rho(x, T_n x, a)\}$$

so that

$$\rho(Tx, S_n x_n, T_n x) = (T_n x, S_n x_n, Tx) = 0.$$

Thus

$$\rho(x, x_n, a) \leqslant \rho(Tx, T_n x, a) + q \max \{\rho(x, x_n, a), \rho(Tx, T_n x, a)\}$$

which gives

$$\rho(x, x_n, a) \leqslant \rho(Tx, T_n x, a)\,(1 - q)^{-1} \to 0 \quad \text{as} \quad n \to \infty.$$

Lemma 2 — If x_n is a fixed point of S_n for each n and $x_n \to x$, then x is a fixed point of T where T is pointwise limit of T_n.

PROOF : As in the proof of Lemma 1,

$$\rho(T_n x, S_n x_n, a) \leqslant q \max \{\rho(x, x_n, a), \rho(x, T_n x, a), \rho(x_n, T_n x, a)\}.$$

Using continuity of ρ and letting $n \to \infty$, one has

$$\rho(Tx, x, a) \leqslant q\rho(x, Tx, a).$$

This gives $\rho(Tx, x, a) = 0$ for all $a \in X$. Hence $Tx = x$, otherwise $\rho(Tx, x, a) \neq 0$ for some a.

PROOF OF THEOREM 2 : Since (i) implies (iii) by Lemma 1 and (iii) implies (i) by Lemma 2, equivalence of (i) and (iii) is established. In a similar way, we can prove the equivalence of (ii) and (iii). Hence the proof.

Remark : When $T_n = S_n$ for each n, we get Theorem B.

3. Some more Generalizations

Following theorems are stated without proof:

Theorem 3 — Let X be a complete 2-metric space with ρ continuous $f : X \to X$ satisfying

(i) $$\rho(fx, fy, a) < \max \{\rho(x, y, a), \rho(x, fx, a), \rho(y, fy, a), \tfrac{1}{2}[\rho(x, fy, a) + \rho(y, fx, a)]\}$$

for each $x, y \in X$ with $x \neq y$;

(ii) there exists a point $x_0 \in X$ for which $\{f^n x_0\}$ contains a subsequence $\{f^{n_i} x_0\}$ converging to $\xi \in X$;

(iii) f and f^2 are continuous at ξ, then ξ is a unique fixed point of f and $f^n x_0 \to \xi$.

Theorem 4 — Let X be a complete 2-metric space with ρ cotinuous

$f, g : X \to X$ satisfying

(i) $$\rho(fx, gy, a) < \max \{\rho(x, y, a), \rho(x, fx, a), \rho(y, gy, a), \tfrac{1}{2}[\rho(x, gy, a) + \rho(y, fx, a)]\}$$

for each $x, y \in X$ with $x \neq y$;

(ii) there exists a point $x_0 \in X$ such that any one of the sequences

$$\{(fg)^n x_0\}, \{g(fg)^n x_0\}, \{(gf)^n x_0\} \text{ and } \{f(gf)^n x_0\}$$

has a cluster point $\xi \in X$;

(iii) f, g, fg and gf are continuous at ξ, then ξ is the unique fixed point of f and g and the sequence of iterates which has a cluster points ξ itself converges to ξ.

Remarks : (a) Proofs of Theorems 3 and 4 are obvious modifications of those of Theorems 1 and 2 of Pal *et al.* (1976). (b) Rhoades (1978) obtained Theorem 3 under different conditions. (c) Theorem 4 is an extension of Theorem 7 of Rhoades (1978). Adopting the technique of Ray and Rhoades (1977), following result for two mappings having a contractive iterate can easily be proved.

Theorem 5 — Let T_1 and T_2 be self-mappings of a complete 2-metric space X which satisfy: "there exists a constant K, $0 < K < 1$ such that there exist positive integers $n(x)$, $m(y)$ such that for all $x, y, a \in X$,

$$\rho(T_1^{n(x)}(x), T_2^{m(y)}(y), a) \leq K \max \{\rho(x, y, a), \rho(x, T_1^{n(x)}(x), a), \rho(y, T_2^{m(y)}(y), a), \tfrac{1}{2}\{\rho(x, T_2^{m(y)}(y), a) + \rho(y, T_1^{n(x)}(x), a)\}\}."$$

Then there exists a unique point u such that $T_1^{n(u)}(u) = T_2^{m(u)}(u)$.

Example

Under the conditions of Theorem 5, T_1 and T_2 need not have a common fixed point. This is shown by the following simple example:

Let $X = \{0, 1\}$ and define ρ to be the ordinary triangular area. Since X contains only two points, ρ is always zero, so the contractive definition is trivially true.

Let $T = T_1 = T_2$ be defined by $T(0) = 1$, $T(1) = 0$, $n(0) = 2$, $m(1) = 1$. Then T_1 and T_2 with $m = n$, have no fixed points, hence no common fixed points.

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