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ON A TRANSFORMATION ASSOCIATED WITH SETS OF n FUNDAMENTAL FORMS OF MINKOWSKIAN HYPERSURFACES

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Let (M^n, L) and (M^n, L^*) be two Finsler spaces whose metric functions $L(x, y)$ and $L^*(x, y)$ are related by $L^*(x, y) = L(x, y) + f_i(x)\, y^i$. Let $(M^{n-1}, \bar{L})$ and $(M^{n-1}, \bar{L}^*)$ be the hypersurfaces of (M^n, L) and (M^n, L^*) respectively represented by the same equation. The purpose of the present paper is to obtain a relation between the pth fundamental forms of tangent Riemannian spaces to $(M^{n-1}, \bar{L})$ and $(M^{n-1}, \bar{L}^*)$.

1. INTRODUCTION

Let (M^n, L) be a n-dimensional Finsler space with fundamental metric function $L(x, y)$. In general $L(x, y)$ is a function of point $x = (x^i)$ and element of support $y = (y^i)$ and positively homogeneous of degree one. If there exist local coordinate systems in which L is independent of x, then (M^n, L) is called the locally Minkowskian space. Let (M^n, L') and (M^n, L^*) be Finsler spaces whose metric functions $L'(x, y)$ and $L^*(x, y)$ are obtained from L by the relations

$$L'(x, y)^2 = L(x, y)^2 + (f_i(x)\, y^i)^2 \quad \ldots(1.1)$$

$$L^*(x, y) = L(x, y) + f_i(x)\, y^i \quad \ldots(1.2)$$

where $f_i(x)$ is a component of a covariant vector which is a function of position alone. These two transformations have been introduced by Matsumoto (1971) which have the geometrical properties as stated below:

Let (M^n, L^*) [resp. (M^n, L')] be the locally Minkowskian n-space obtained from a locally Minkowskian n-space (M^n, L) by the transformation (1.2) [resp. (1.1)]. If the tangent Riemannian n-space (M^n_x, g_x) to (M^n, L) is of imbedding class r, then the tangent Riemannian n-space (M^n_x, g^*_x) to (M^n, L^*) [resp. (M^n_x, g'_x) to (M^n, L')] is of imbedding class at most $r + 2$ (resp. $r + 1$).

If $L(x, y)$ will be the metric function of Riemannian manifold then the function $L^*(x, y)$ reduces to the metric function of Randers space. Such a space was first introduced by Randers (1941) from the stand point of general relativity and was applied to the theory of electron microscope by Ingardan (1957).

The n-fundamental forms of a Riemannian hypersurface have been defined and their properties have been studied by Rund (1971a). Prasad (1978) has obtained the relation in n-fundamental forms of tangent Riemannian hypersurfaces of (M^n, L) and (M^n, L'). The purpose of this paper is to obtain the relation in n-fundamental forms of tangent Riemannian hypersurfaces of (M^n, L) and (M^n, L^*).

2. Hypersurface of (M^n, L)

Let $(M^{n-1}, \bar{L})$ be a hypersurface of (M^n, L) given by equation

$$x^i = x^i(u^\alpha). \qquad \text{...(2.1)}$$

Let us suppose that the functions (2.1) are at least of class C^3 in u^α and the projection factor $B^j_\alpha = \dfrac{\partial x^j}{\partial u^\alpha}$ are such that their matrix has maximal rank $(n-1)$. The fundamental metric function $\bar{L}(u, v)$ of the hypersurface is given by

$$\bar{L}(u^\alpha, v^\alpha) = L(x^i(u^\alpha), B^i_\alpha v^\alpha)$$

where v^α is the element of support for the hypersurface for which

$$y^i = B^i_\alpha v^\alpha. \qquad \text{...(2.2)}$$

If $g_{hj}(x, y)$ denotes the metric tensor of (M^n, L) the induced metric tensor of $(M^{n-1}, \bar{L})$ is given by

$$g_{\alpha\beta}(u, v) = g_{hj}(x, y)\, B^h_\alpha B^j_\beta\,. \qquad \text{...(2.3)}$$

The inverse of (2.3) is denoted by $g^{\alpha\beta}(u, v)$ by means of which we define the quantities

$$B^\alpha_i\,(u, v) = g^{\alpha\beta}(u, v)\, g_{ij}(x, y)\, B^j_\beta\,. \qquad \text{...(2.4)}$$

The unit normal vector $N^j(x, y)$ of $(M^{n-1}, \bar{L})$ is determined by the relations

$$g_{hj}(x, y)\, B^h_\beta\, N^j(x, y) = 0,\; g_{hj}(x, y)\, N^h(x, y)\, N^j(x, y) = 1. \qquad \text{...(2.5)}$$

We have the following identity from (2.3), (2.4) and (2.5)

$$B^\alpha_j B^j_\beta = \delta^\alpha_\beta,\; B^j_\alpha B^\alpha_h + N^j N_h = \delta^j_h \qquad \text{...(2.6)}$$

where $N_h = g_{hl}(x, y)\, N^l$. If $C_{hjk}(x, y)$ denotes the Cartan tensor of (M^n, L), the induced Cartan tensor $C_{\alpha\beta\gamma}(u, v)$ of $(M^{n-1}, \bar{L})$ is given by

$$C_{\alpha\beta\gamma}(u, v) = C_{hjk}(x, y)\, B^h_\alpha B^j_\beta B^k_\gamma \qquad \text{...(2.7)}$$

from which we obtain

$$C^{\alpha}_{\beta\gamma} = B^{\alpha}_{i}\, C^{i}_{jk}\, B^{j}_{\beta}\, B^{k}_{\gamma}. \qquad \ldots(2.8)$$

The mixed v-covariant derivative of the projection factors B^{i}_{β} is defined as

$$Z^{i}_{\beta\gamma} = B^{i}_{\beta\,|\,\gamma} = -\, B^{i}_{\alpha}\, C^{\alpha}_{\beta\gamma} + C^{i}_{hk}\, B^{h}_{\beta}\, B^{k}_{\gamma}. \qquad \ldots(2.9)$$

From (2.8) it follows that $Z^{i}_{\beta\gamma}$ is normal to $(M^{n-1}, \bar{L})$. Therefore, we may write

$$Z^{i}_{\beta\gamma} = \omega_{\beta\gamma} N^{i}. \qquad \ldots(2.10)$$

From (2.9) it is clear that $\omega_{\beta\gamma}$ is symmetric in β and γ.

Now the tangent vector space M^{n-1}_{x} to M^{n-1} at every point x^{i} $(= u^{\alpha})$ of the hypersurface is considered as the Riemannian space $(M^{n-1}_{x}, \bar{g}_{x})$ with the Riemannian metric $\bar{g}_{x} = g_{\alpha\beta}(u, v)\, dv^{\alpha} dv^{\beta}$. The components of the Cartan tensor $C^{\alpha}_{\beta\gamma}$ will be the Christoffel symbols associated with $\bar{g}_{x}$. If M^{n}_{x} is the tangent vector space to M^{n} at $x^{i}(= u^{\alpha})$ then $(M^{n-1}_{x}, \bar{g}_{x})$ will be the hypersurface of (M^{n}_{x}, g_{x}) given by eqn. (2.2) where $g_{x} = g_{ij}(x, y)\, dy^{i} dy^{j}$ is the Riemannian metric on M^{n}_{x}. The factor of proportionality of eqn. (2.10) will be considered as the coefficients of second fundamental form of tangent Riemannian space $(M^{n-1}_{x}, \bar{g}_{x})$.

In general the coefficients of the pth fundamental form of $(M^{n-1}, \bar{g}_{x})$ are defined as (Rund 1971b)

$$\left.\begin{aligned} C_{(1)\alpha\beta} &= g_{\alpha\beta} \\ C_{(2)\alpha\beta} &= \omega_{\alpha\beta} \\ C_{(p)\alpha\beta} &= C_{(p-1)\alpha\epsilon}\,\omega^{\epsilon}_{\beta} \;\; (2 \leqslant p \leqslant n) \end{aligned}\right\} \qquad \ldots(2.11)$$

where

$$\omega^{\epsilon}_{\beta} = g^{\alpha\epsilon}\omega_{\alpha\beta}.$$

3. Hypersurface of (M^{n}, L^{*})

Let $(M^{n-1}, \bar{L}^{*})$ be a hypersurface of (M^{n}, L^{*}) given by the same eqn. (2.1). We shall denote the quantities of $(M^{n-1}, \bar{L}^{*})$ and (M^{n}, L^{*}) by star letters. From (1.1) it follows that the metric tensors and Cartan tensors of (M^n, L) and (M^n, L^*) are related by (Matsumoto 1971)

$$g^*_{ij} = L^*L^{-1}g_{ij} - FL^{-3}y_iy_j + L^{-1}(y_if_j + y_jf_i) + f_if_j \quad ...(3.1)$$

$$g^{*ij} = LL^{*-1}g^{ij} - LL^{*-2}(y^if^j + y^jf^i) + (Lf^2 + F)L^{*-3}y^iy^j \quad ...(3.2)$$

$$C^*_{ijk} = L^*L^{-1}C_{ijk} - \tfrac{1}{2}L^{-1}(a_{ij}b_k + a_{jk}b_i + a_{ki}b_j) \quad ...(3.3)$$

$$\left.\begin{aligned} C^{*j}_{ik} &= C^j_{ik} - \tfrac{1}{2}L^{*-1}(a^j_i b_k + a^j_k b_i + a_{ik}b^j) \\ &\quad - (L^{*-1}C_{lik}f^l - \tfrac{1}{2}L^{*-2}(b_lf^la_{ik} - 2b_ib_k))\,y^j \end{aligned}\right\} \quad ...(3.4)$$

where

$$\left.\begin{aligned} &\text{(a)} \quad f^i = g^{ij}f_j,\ f^2 = g^{ij}f_if_j,\ y_i = g_{ij}(x, y)\,y^j \\ &\text{(b)} \quad a_{ij} = g_{ij} - L^{-2}y_iy_j,\ b_i = FL^{-2}y_i - f_i,\ F = f_iy^i. \end{aligned}\right\} \quad ...(3.5)$$

Relations (3.1), (3.2) and (2.3) yield

$$g^*_{\alpha\beta} = L^*L^{-1}g_{\alpha\beta} - FL^{-3}v_\alpha v_\beta + L^{-1}(v_\alpha f_\beta + v_\beta f_\alpha) + f_\alpha f_\beta \quad ...(3.6)$$

$$g^{*\alpha\beta} = LL^{*-1}g^{\alpha\beta} - LL^{*-2}(v^\alpha f^\beta + v^\beta f^\alpha) + (Lb^2 + F)L^{*-3}v^\alpha v^\beta \quad ...(3.7)$$

where

$$\text{(a)} \quad f_\alpha = f_iB^i_\alpha,\ \text{(b)}\ b^2 = g^{\alpha\beta}f_\alpha f_\beta,\ \text{(c)}\ f^\alpha = g^{\alpha\beta}f_\beta,\ \text{(d)}\ v_\alpha = g_{\alpha\beta}v^\beta. \quad ...(3.8)$$

From (3.6) and (3.7), we have

$$C^*_{\alpha\beta\gamma} = L^*L^{-1}C_{ijk} - \tfrac{1}{2}L^{-1}(a_{\alpha\beta}b_\gamma + a_{\beta\gamma}b_\alpha + a_{\gamma\alpha}b_\beta) \quad ...(3.9)$$

$$\begin{aligned} C^{*\alpha}_{\beta\gamma} &= C^\alpha_{\beta\gamma} - \tfrac{1}{2}L^{*-1}(a^\alpha_\beta\, b_\gamma + \delta^\alpha_\gamma b_\beta + a_{\beta\gamma}b^\alpha) \\ &\quad - (L^{*-1}C_{\delta\beta\gamma}f^\delta - \tfrac{1}{2}L^{*-2}(b_\delta f^\delta a_{\beta\gamma} - 2b_\beta b_\gamma))\,v^\alpha \end{aligned} \quad ...(3.10)$$

where

$$a_{\beta\gamma} = g_{\beta\gamma} - L^{-2}v_\beta v_\gamma,\ b_\beta = FL^{-2}v_\beta - f_\beta. \quad ...(3.11)$$

In general the vector f^i is not tangential to the hypersurface. However, we may write

$$f^i = f^\alpha B^i_\alpha + \lambda N^i \quad ...(3.12)$$

where $\lambda = f^iN_i = f_iN^i$. From (3.5*b*), (3.8*b*) and (3.12), we get

$$f^2 = b^2 + \lambda^2,\ f^\alpha = B^\alpha_i f^i. \quad ...(3.13)$$

It is to be noted that if N^j is a unit normal vector to $(M^{n-1}, \bar{L})$, then it is not normal to $(M^{n-1}, \bar{L}^*)$. We may write

$$N^j = m^\alpha B^j_\alpha + kN^{*j}. \qquad ...(3.14)$$

To obtain m^α and k we use (3.1), (2.5) and (3.8). Thus we get

$$g^*_{hj} B^h_\alpha N^j = \lambda(p_\alpha + f_\alpha) \qquad ...(3.15)$$

$$g^*_{hj} N^h N^j = L^* L^{-1} + \lambda^2 \qquad ...(3.16)$$

where $p_\alpha = g_{\alpha\beta} v^\beta / L$ is unit vector in $(M^{n-1}, \bar{L})$ along the element of support. From (2.3), (3.14), (3.15) and (3.16) it follows that

$$g^*_{\alpha\beta} m^\beta = \lambda(p_\alpha + f_\alpha) \qquad ...(3.17)$$

$$k^2 = L^* L^{-1} + \lambda^2 - \lambda(p_\alpha + f_\alpha) m^\alpha. \qquad ...(3.18)$$

If f^i is tangential to the hypersurface $(M^{n-1}, \bar{L})$, then $\lambda = 0$, $m^\alpha = 0$ and $k = \sqrt{L^* L^{-1}}$. From (3.14), it follows that $N^{*j} = L^{*-1/2} L^{1/2} N^j$.

Hence we have the following theorem:

Theorem 1 — Let (M^n, L^*) be a Finsler space obtained from a Finsler space (M^n, L) by the transformation (1.2). If $(M^{n-1}, \bar{L}^*)$ and $(M^{n-1}, \bar{L})$ are hypersurfaces of these spaces and f^i is tangential to the hypersurface $(M^{n-1}, \bar{L})$ then the vector normal to $(M^{n-1}, \bar{L})$ is also normal to $(M^{n-1}, \bar{L}^*)$.

Now we shall establish the following:

Theorem 2 — Let (M^n, L^*) be the locally Minkowskian space obtained from a locally Minkowskian space (M^n, L) by the transformation (1.2). Let $(M^{n-1}, \bar{L}^*)$ and $(M^{n-1}, \bar{L})$ be hypersurfaces of (M^n, L^*) and (M^n, L) respectively. If f^i is tangential to the hypersurface $(M^{n-1}, \bar{L})$ and (M^n_x, g_x), (M^n_x, g^*_x), $(M^{n-1}_x, \bar{g}_x)$, $(M^{n-1}_x, \bar{g}^*_x)$ are tangent Riemannian spaces to (M^n, L), (M^n, L^*), $(M^{n-1}, \bar{L})$, $(M^{n-1}, \bar{L}^*)$ respectively, then we have the following:

(i) Second fundamental forms of $(M^{n-1}_x, \bar{g}_x)$ and $(M^{n-1}_x, \bar{g}^*_x)$ are proportional.

(ii) Every asymptotic direction of $(M_x^{n-1}, \bar{g}_x)$ is asymptotic direction of $(M_x^{n-1}, \bar{g}_x^*)$.

(iii) The pth fundamental tensors of $(M_x^{n-1}, \bar{g}_x)$ and $(M_x^{n-1}, \bar{g}_x^*)$ are related by

$$C^*_{(p)\alpha\beta} = L^{1/2(p-3)}L^{*1/2(3-p)}\left[C_{(p)\alpha\beta} - \sum_{m=2}^{p-1}(X_{(m)\beta}P_{(p+1-m)\alpha} - Y_{(m)\beta}Q_{(p+1-m)\alpha})\right] \quad (3 \leqslant p \leqslant n) \qquad \ldots(3.19)$$

where

(a) $X_{(m)\beta} = L^{*-1/2}C_{(m)\alpha\beta}f^\alpha \qquad (2 \leqslant m \leqslant n-1)$

(b) $Y_{(m)\beta} = L^{*-1/2}C_{(m)\alpha\beta}\nu^\alpha \qquad (2 \leqslant m \leqslant n-1)$, (c) $P_{(2)\alpha} = Y_{(2)\alpha}$

(d) $P_{(p)\alpha} = Y_{(p)\alpha} - \sum_{m=2}^{p-1}(C_{(m)}P_{(p+1-m)\alpha} - Y_{(m)}Q_{(p+1-m)\alpha}) \quad (3 \leqslant p \leqslant n-1)$

(e) $Q_{(2)\alpha} = X_{(2)\alpha} - GY_{(2)\alpha}$

(f) $Q_{(p)\alpha} = X_{(p)\alpha} - GY_{(p)\alpha} - \sum_{m=2}^{p-1}(X_{(m)} - GC_{(m)})\,P_{(p+1-m)\alpha} - \sum_{m=2}^{p-1}(C_{(m)} - GY_{(m)})\,Q_{(p+1-m)\alpha} \quad (3 \leqslant p \leqslant n-1)$

(g) $G = \dfrac{Lb^2 + F}{LL^*}$, (h) $X_{(m)} = L^{*-1/2}X_{(m)\alpha}f^\alpha \qquad (2 \leqslant m \leqslant n-1)$

(i) $Y_{(m)} = L^{*-1/2}Y_{(m)\alpha}\nu^\alpha \qquad (2 \leqslant m \leqslant n-1)$

(j) $C_{(m)} = L^{*-1}C_{(m)\alpha\beta}f^\alpha\nu^\beta \qquad (2 \leqslant m \leqslant n-1). \qquad \ldots(3.20)$

PROOF: (i) If $Z^{*j}_{\beta\gamma}$ denote the tensor derivative of B^j_β in tangent Riemannian hypersurface $(M_x^{n-1}, \bar{g}_x^*)$ of the tangent Riemannian space (M_x^n, g_x^*), then from (2.9), (3.4), (3.10) and (3.12) we have

$$Z^{*j}_{\beta\gamma} = Z^j_{\beta\gamma} + \tfrac{1}{2}\lambda L^{*-1}a_{\beta\gamma}N^j - \lambda L^{*-1}C_{lik}N^l B^i_\beta B^k_\gamma y^j - \tfrac{1}{2}\lambda^2 L^{*-2}a_{\beta\gamma}y^j. \qquad \ldots(3.21)$$

In view of (3.14) and (2.10) the relation (3.21) yield

$$m^\alpha\omega^*_{\beta\gamma} = k\lambda L^{*-1}\nu^\alpha(C_{lik}B^i_\beta B^k_\gamma N^l + \tfrac{1}{2}\lambda L^{*-1}a_{\beta\gamma}) \qquad \ldots(3.22)$$

$$\omega^*_{\beta\gamma} = k\omega_{\beta\gamma} + \tfrac{1}{2}\lambda kL^{*-1}a_{\beta\gamma} \qquad \ldots(3.23)$$

From (3.12) it follows that if f^i is tangential to $(M^{n-1}, \bar{L})$, then $\lambda = 0$ and hence from (3.23) we have $\omega^*_{\beta\gamma} = k\omega_{\beta\gamma}$. This proves (i).

(ii) A direction t^α for which $\omega_{\beta\gamma}t^\beta t^\gamma = 0$ is said to be an asymptotic direction. In view of this definition and (i) we get (ii).

(iii) The validity of the relation (3.19) is established by induction. From (3.7), (3.20) and (3.23) for $\lambda = 0$, we have

$$\omega^{*\epsilon}_{\beta} = \omega^*_{\alpha\beta}\, g^{*\alpha\epsilon} = L^{1/2}L^{*-1/2}\,[\omega^{\epsilon}_{\beta} - L^{*-1/2}(X_{(2)\beta} - GY_{(2)\beta})\,\nu^\epsilon - L^{*-1/2}Y_{(2)\beta}f^\epsilon]. \qquad \text{...(3.24)}$$

The relations (2.11) and (3.24) yield

$$C^*_{(3)\alpha\beta} = C^*_{(2)\alpha\epsilon}\,\omega^{*\epsilon}_{\beta} = C_{(3)\alpha\beta} - X_{(2)\beta}Y_{(2)\alpha} - Y_{(2)\beta}Q_{(2)\alpha}. \qquad \text{...(3.25)}$$

From (3.25) and (3.20) it is evident that (3.19) holds for $p = 3$. For a given fixed value of the integer s with $3 \leqslant s \leqslant n-1$, we have

$$C^*_{(s+1)\alpha\beta} = C^*_{(s)\alpha\epsilon}\,\omega^{*\epsilon}_{\beta}\,. \qquad \text{...(3.26)}$$

Now let us suppose that (3.19) is valid for $s = 3, 4, 5, \ldots, p$ so that we can write (3.26) in the form

$$C^*_{(s+1)\alpha\beta} = L^{1/2(s-3)}L^{*1/2(3-s)}\,[C_{(s)\alpha\epsilon} - \sum_{m=2}^{s-1}(X_{(m)\epsilon}P_{(s+1-m)\alpha} - Y_{(m)\epsilon}Q_{(s+1-m)\alpha})]\,\omega^{*\epsilon}_{\beta}$$

which in view of (3.24), (3.20) gives

$$\begin{aligned} C^*_{(s+1)\alpha\beta} &= L^{1/2(s-2)}L^{*1/2(2-s)}\,[C_{(s+1)\alpha\beta} - \sum_{m=2}^{s-1}(X_{(m+1)\beta}P_{(s+1-m)\alpha} \\ &\qquad - Y_{(m+1)\beta}Q_{(s+1-m)\alpha}) - X_{(2)\beta}P_{(s)\alpha} - Y_{(2)\beta}Q_{(s)\alpha}] \\ &= L^{1/2(s-2)}L^{*1/2(2-s)}\,[C_{(s+1)\alpha\beta} - \sum_{m=2}^{s}(X_{(m)\beta}P_{(s+2-m)\alpha} \\ &\qquad - Y_{(m)\beta}Q_{(s+2-m)\alpha})]. \end{aligned}$$

This shows that (3.19) is valid for $p = s + 1$ which completes the proof of (iii).

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