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ON | c, 1 | -SUMMABILITY OF ULTRASPHERICAL SERIES

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Beohar (1967) proved the | c, δ | -summability of the Laplace series when $\delta > 1$. Here two new theorems are proved on the absolute | c, 1 | -summability of ultraspherical series defined on a sphere S.

§1. Let $f(\theta, \varphi)$ be a function defined for the range $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \varphi \leqslant 2\pi$, the ultraspherical series corresponding to $f(\theta, \varphi)$ on the sphere S is

$$f(\theta, \varphi) \sim \frac{1}{2\pi} \sum_{n=0}^{\infty} (n + \lambda) \iint_S \frac{f(\theta', \varphi') P_n^{(\lambda)} (\cos \omega) \sin \theta' d\theta' d\varphi'}{[\sin^2 \theta' \sin^2 (\varphi - \varphi')]^{(1-2\lambda)/2}} \quad \ldots(1.1)$$

where $\cos \omega = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\varphi - \varphi')$ and ultraspherical polynomials $P_n^{(\lambda)}(x)$ is defined by the following

$$(1 - 2xt + t^2)^{-\lambda} = \sum_{n=0}^{\infty} t^n P_n^{(\lambda)} (x), \ \lambda > 0.$$

The Laplace series is a particular case of the series (1.1) for $\lambda = \frac{1}{2}$ and further (1.1) reduces to the trigonometric series in the limiting case as

$$\lim_{\lambda \to 0} \frac{1}{\lambda} P_n^{(\lambda)} (\cos \theta) = \frac{2}{n} \cos n\theta, \quad n \geqslant 1. \quad \ldots(1.2)$$

A generalized mean value of $f(\theta, \varphi)$ on the sphere has been defined by Kogbetliantz (1924) as follows:

$$f(\omega) = \frac{1}{2\pi(\sin \omega)^{2\lambda}} \int_{c_\omega} \frac{f(\theta', \varphi') \, ds'}{[\sin^2 \theta' \sin^2 (\varphi - \varphi')]^{(1-2\lambda)/2}} \quad \ldots(1.3)$$

where the integral is taken along the small circle c whose centre is (θ, φ) on the sphere S and whose curvilinear radius is ω.

We write

$$\varphi(\omega) = f(\omega) \, . \, (\sin \omega)^{2\lambda - 1} \quad \ldots(1.4)$$

$$\Phi_p(x) = \frac{1}{\Gamma(p)} \int_0^x (x - t)^{p-1} \varphi(t) \, dt \quad \ldots(1.5)$$

$$\Phi_0(x) = \varphi(x),\ \varphi_p(x) = \Gamma(p+1)\, x^{-p}\Phi_p(x),\ \ p \geqslant 0 \qquad \text{...(1.6)}$$

$$\Phi_p(x) = \frac{d}{dx}\,\Phi_{p+1}(x). \qquad \text{...(1.7)}$$

We prove the following:

Theorem 1 — If $\varphi(\omega)$ is of bounded variation in (η, π), where

$$\eta = \frac{\mu}{n^{\Delta}},\ \frac{1-\lambda}{\lambda} > \Delta > 0,\ 0 < \lambda < 1 \qquad \text{...(1.8)}$$

μ is a large constant, and if

$$\Phi_1(t) \equiv \int_0^t |\,\varphi(\omega)\,|\, d\omega = O\left\{\frac{t^{\alpha}}{(\log 1/t)^{1+\epsilon}}\right\}_{\epsilon>0} \qquad \text{...(1.9)}$$

$\alpha = \dfrac{2\lambda + 1 - \Delta}{\Delta}$, as $t \to 0$, then the series (1.1) is summable $|\, c, 1\,|$.

Theorem 2 — If $\varphi(\omega)$ is of bounded variation in (η, π), where $\eta = \mu/n^{\Delta}$; Δ is a positive real number less than unity satisfying the relation

$$\frac{1-\lambda}{\lambda} > \Delta > \frac{1+2\lambda}{3+\beta-\alpha},\ (0 \leqslant \alpha \leqslant \beta < 1) \qquad \text{...(1.10)}$$

μ is a large constant, and if

$$\Phi_{\alpha}(t) = O(t^{1+\beta}),\ t \to 0 \qquad \text{...(1.11)}$$

then the series (1.1) is summable $|\, c, 1\,|$.

§2. To prove the theorems we require the following lemmas.

Lemma 1 (Szegö 1967, p. 171) — We have, for $\lambda > 0$

$$P_n^{(\lambda)}(\cos\theta) = \begin{cases} \theta^{-\lambda} O(n^{\lambda-1}), & c/n \leqslant \theta \leqslant \pi/2 \\ O(n^{2\lambda-1}) & ,\ 0 \leqslant \theta \leqslant c/n \end{cases} \qquad \text{...(2.1)}$$

and $\qquad (\sin\theta)^{\lambda}\, |\, P_n^{(\lambda)}(\cos\theta)\,| < 2^{1-\lambda}\{\Gamma(\lambda)\}^{-1} n^{\lambda-1},\ 0 < \lambda < 1,\ 0 \leqslant \theta \leqslant \pi.$

Lemma 2 (Szegö 1967, p. 84) — For $n \geqslant 0$, we have

$$\frac{d}{dx}\{P_n^{(\lambda)}(x)\} = 2\lambda P_{n-1}^{(\lambda+1)}(x),\ \ P_{-1}^{(\lambda)}(x) = 0. \qquad \text{...(2.2)}$$

Lemma 3 (Yadav 1977, p. 539) — Let s_n be the nth partial sum of an infinite series $\Sigma\, a_m$ and let

$$\sum_{n=1}^{\infty} \frac{|s_n|}{n^\delta} < \infty, \; 0 \leqslant \delta \leqslant 1 \qquad \ldots(2.3)$$

then the series $\Sigma\, a_m$ is summable $|\, c, \delta \,|$.

Proof of the Theorem 1 — Let s_n be the nth partial sum of the series (1.1), then we have (Szegö 1967, p. 84)

$$s_n = \frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+\lambda)} \int_0^\pi f(\omega) \sum_{k=0}^{n} (k+\lambda)\, P_k^{(\lambda)}(\cos\omega)\,(\sin\omega)^{2\lambda}\, d\omega$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+\lambda)} \int_0^\pi f(\omega) \left[\frac{d}{dx}\{P_{n+1}^{(\lambda)}(x) + P_n^{(\lambda)}(x)\}\right]_{x=\cos\omega} \times (\sin\omega)^{2\lambda}\, d\omega$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+\lambda)} \int_0^\pi \varphi(\omega) \frac{d}{d\omega}\{P_{n+1}^{(\lambda)}(\cos\omega) + P_n^{(\lambda)}(\cos\omega)\}\, d\omega$$

$$= s_n^1 + s_n^2, \text{ says.}$$

Now

$$s_n^1 = \frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+\lambda)} \left[\int_0^\eta + \int_\eta^\pi\right]$$

$$= J_1 + J_2, \text{ say.}$$

But

$$J_1 = O(n^{2\lambda+1}) \int_0^\eta \omega\, |\varphi(\omega)|\, d\omega$$

$$= O(n^{2\lambda+1})\, \eta\eta^\alpha \frac{1}{(\log n)^{1+\epsilon}}$$

$$= O(n^{2\lambda+1}) \frac{n^{-\Delta(\alpha+1)}}{(\log n)^{1+\epsilon}}$$

$$= O\left[\frac{1}{(\log n)^{1+\epsilon}}\right]_{\epsilon>0}$$

and

$$J_2 = \frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+\lambda)} \int_{\eta}^{\pi} \varphi(\omega) \frac{d}{d\omega} \{P_{n+1}^{(\lambda)}(\cos\omega)\}\, d\omega$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+\lambda)} \left[\left\{ \varphi(\omega)\, P_{n+1}^{(\lambda)}(\cos\omega) \right\}_{\eta}^{\pi} - \int_{\eta}^{\pi} P_{n+1}^{(\lambda)}(\cos\omega)\, d\varphi(\omega) \right]$$

$$= O(n^{\lambda-1})\, \eta^{-\lambda}$$

$$= O(n^{\lambda-1+\Delta\lambda}).$$

Other parts of the sum are treated similarly and consequently in the light of Lemma 3, Theorem 1 is proved.

N.B. If $\alpha > \dfrac{2\lambda+1-\Delta}{\Delta}$, we can replace (1.9) by the following condition,

$$\Phi_1(t) = O(t^{\alpha}) \qquad \ldots(2.4)$$

In this case $J_1 = O(n^{2\lambda+1-\Delta(\alpha+1)})$, and hence $\sum \dfrac{|J_1|}{n} < \infty$.

Proof of Theorem 2 — To prove this theorem it is sufficient to calculate the order of the integral

$$I \equiv \int_0^{\eta} f(\omega) \left\{ \frac{d}{d\omega} P_{n+1}^{(\lambda)}(\cos\omega) \right\} (\sin\omega)^{2\lambda-1}\, d\omega$$

under the hypothesis of the theorem.

We have

$$I = \int_0^{\eta} \varphi(\omega) \left\{ \frac{d}{d\omega} P_{n+1}^{(\lambda)}(\cos\omega) \right\} d\omega$$

$$= \Big[2\lambda \sin\omega\, P_n^{(\lambda+1)}(\cos\omega)\, \Phi_1(\omega) \Big]_0^{\eta}$$

$$- 2\lambda \int_0^{\eta} \Phi_1(\omega) \frac{d}{d\omega} \{\sin\omega\, P_{n+1}^{(\lambda+1)}(\cos\omega)\}\, d\omega$$

$$= I_1 - I_2, \text{ say.}$$

But,

$$I_1 = O(n^{2\lambda+1})\,\eta\,\eta^{2+\beta-\alpha}$$
$$= O\{n^{2\lambda+1-\Delta(3+\beta-\alpha)}\}$$

$$I_2 = 2\lambda \int_0^{\eta} \frac{d}{d\omega}\{\sin\omega\, P_n^{(\lambda+1)}(\cos\omega)\}\left[\frac{1}{\Gamma(1-\beta)}\int_0^{\omega}(\omega-u)^{-\alpha}\Phi_\alpha(u)\,du\right]d\omega$$

$$= \frac{2\lambda}{\Gamma(1-\beta)}\int_0^{\eta}\Phi_\alpha(u)\left[\int_u^{\eta}\frac{d}{d\omega}\{\sin\omega\, P_n^{(\lambda+1)}(\cos\omega)\}(\omega-u)^{-\alpha}\,d\omega\right]du$$

$$= \left[\int_0^{1/n} + \int_{1/n}^{\eta}\right]\Phi_\alpha(u)\,F(\eta, u)\,du$$

$$= I_{2\cdot1} + I_{2\cdot2}, \text{ say.}$$

We approximate $F(\eta, u)$ for the first integral $I_{2\cdot1}$ i.e. when u is ranging from 0 to $1/n$.

$$F(\eta, u) = \int_u^{\eta}(\omega-u)^{-\alpha}\frac{d}{d\omega}\{\sin\omega\, P_n^{(\lambda+1)}(\cos\omega)\}\,d\omega$$

$$= \left[\int_u^{2u} + \int_{2u}^{\eta}\right](\omega-u)^{-\alpha}\frac{d}{d\omega}\{\sin\omega\, P_n^{(\lambda+1)}(\cos\omega)\}\,d\omega$$

$$= K\int_u^{2u}(\omega-u)^{-\alpha}\Big\{\cos\omega\, P_n^{(\lambda+1)}(\cos\omega) + \sin\omega\,\frac{d}{d\omega}P_n^{(\lambda+1)}(\cos\omega)\Big\}\,d\omega$$
$$+ O(u^{-\alpha})\int_{2u}^{\xi}\frac{d}{d\omega}\{\sin\omega P_n^{(\lambda+1)}(\cos\omega)\}\,d\omega,\quad 2u \leqslant \xi \leqslant \eta$$

$$= O(n^{2\lambda+1})\,u^{1-\alpha} + O(n^{2\lambda+3})\,u^{3-\alpha} + O(u^{-\alpha})\Big[\sin\omega\, P_n^{(\lambda+1)}(\cos\omega)\Big]_{2u}^{\xi}$$

$$= O(n^{2\lambda+1})\,u^{1-\alpha} + O(n^{2\lambda+3})\,u^{3-\alpha} + O(n^{2\lambda+1-\Delta})\,u^{-\alpha}.$$

Again, we have the following order estimate for $F(\eta, u)$ when u ranges from $1/n$ to η.

$$F(\eta, u) = \left[\int_{u}^{u+(1/n)} + \int_{u+(1/n)}^{\eta} \right] (\omega - u)^{-\alpha} \frac{d}{d\omega} \{\sin \omega \, P_n^{(\lambda+1)} (\cos \omega)\} \, d\omega$$

$$= O(n^{\lambda})\, n^{\alpha-1} u^{-\lambda-1} + u^{2-\lambda-2} n^{\lambda+1}\, O(n^{\alpha-1})$$

$$+ O(n^{\alpha}) \max \left[\sin \omega \, P_n^{(\lambda+1)} (\cos \omega) \right]_{u+(1/n)}^{\xi} ; \; u + (1/n) \leqslant \xi \leqslant \eta$$

$$= O(n^{\lambda+\alpha-1})\, u^{-\lambda-1} + O(n^{\lambda+\alpha})\, u^{-\lambda} + O(n^{\alpha+\lambda})\, \eta u^{-\lambda-1}.$$

If $u + (1/n)$ is not less than η we do not need to break the integral into two parts.

Now

$$I_{2\cdot1} = \int_0^{1/n} O(u^{2+\beta-\alpha})\, n^{2\lambda+1}\, du + \int_0^{1/n} O(u^{4+\beta-\alpha})\, n^{2\lambda+3}\, du$$

$$+ \int_0^{1/n} O(u^{1+\beta-\alpha})\, n^{2\lambda+1-\Delta}\, dt$$

$$= O(n^{-(3+\beta-\alpha)+2\lambda+1}) + O(n^{-(5+\beta-\alpha)+2\lambda+3}) + O(n^{-(2+\beta-\alpha)+2\lambda+1-\Delta})$$

$$= O(n^{-(2+\beta-\alpha)+2\lambda}) + O(n^{2\lambda+1-\Delta(3+\beta-\alpha)+\Delta(2+\beta-\alpha)-(2+\beta-\alpha)})$$

and

$$I_{2\cdot2} = O(n^{\lambda+\alpha-1}) \int_{1/n}^{\eta} u^{\beta-\lambda}\, du + O(n^{\lambda+\alpha}) \int_{1/n}^{\eta} u^{1+\beta-\lambda}\, du$$

$$+ O(n^{\alpha+\lambda-\Delta}) \int_{1/n}^{\eta} u^{1+\beta-\lambda-1}\, du$$

$$= O(n^{\lambda+\alpha-1-\Delta(1+\beta-\lambda)}) + O(n^{2\lambda+1-\Delta(3+\beta-\alpha)-(1+\lambda-\alpha)+\Delta(1+\lambda-\alpha)})$$

$$+ O\{n^{2\lambda+1-\Delta(3+\beta-\alpha)-1-\lambda+\alpha-\Delta(-1+\alpha-\lambda)}\}.$$

Now, applying Lemma 3, we see that the proof of Theorem 2 is complete.

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