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PSEUDO FUNCTIONS AND SINGULAR PRODUCTS OF DISTRIBUTIONS

A. K. TIWARI

Department of Mathematics, Regional College of Engineering, Nagpur 440011

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A pseudo distribution is defined by extracting a finite part from a divergen integral. The following pseudo distributions are obtained :

$$P_f\{(x_+^r) \cdot \delta^{r+2m-1}\} = \frac{(-1)^r (r+2m-1)!}{2(2m-1)!} \delta^{2m-1}(x)$$

$$P_f\{(x_+^r) \cdot \delta^{r+2m}\} = \frac{(-1)^r (r+2m)!}{2(2m)!} \delta^{2m}(x)$$

$$P_f\{(x_-^r) \cdot \delta^{r+2m-1}\} = \frac{(r+2m-1)!}{2(2m-1)!} \delta^{2m-1}(x)$$

$$P_f\{(x_-^r) \cdot \delta^{r+2m}\} = \frac{(r+2m)!}{2(2m)!} \delta^{2m}(x)$$

for $r = 0, 1, 2, \ldots$ and $m = 1, 2, \ldots$.

1. INTRODUCTION

The product of two distributions f and g on the open interval (a, b), where $-\infty \leqslant a < b \leqslant \infty$, has been defined by Fisher (1971) as the limit of the sequence $\{fn \cdot gn\}$, provided this sequence is regular on (a, b), where $fn = f * \delta_n$, $gn = g * \delta_n$, $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$ and ρ is a fixed infinitely differentiable function having the following properties :

$$\rho(x) = 0 \quad \text{for } |x| \geqslant 1 \qquad \ldots(1)$$

$$\rho(x) \geqslant 0 \qquad \ldots(2)$$

$$\rho(x) = \rho(-x) \qquad \ldots(3)$$

$$\int_{-1}^{1} \rho(x)dx = 1. \qquad \ldots(4)$$

It is obvious that $\{\delta_n\}$ is a regular sequence converging to the Dirac delta-function $\delta(x)$.

If the sequence $\{fn \cdot gn\}$ is not regular so that the product $f \cdot g$ is not defined we say that the product $f \cdot g$ is singular or divergent.

In the following we denote the space of infinitely differentiable test functions ϕ with compact support by D. The subspace $D\{i_1, i_2, i_3, \ldots, i_s\}$ of D is the space of all test functions ϕ for which

$$\phi^{i_1}(0) = \phi^{i_2}(0) = \ldots = \phi^{i_s}(0) = 0$$

The products $(x_+^r) \cdot \delta^{r+1}$, $(x_+^r) \cdot \delta^{r+2}$, $(x_+^r) \cdot \delta^{r+2m-1}$ and $(x_+^r) \cdot \delta^{r+2m}$ have been defined by Fisher (1973, 1972b). The products $(x_-^r) \cdot \delta^{r+2m-1}$ and $(x_-^r) \cdot \delta^{r+2m}$ have been also obtained by Fisher (1972b) as given below :

$$(x_-^r) \cdot \delta^{r+2m-1}(x) = \frac{-(r+2m-1)!}{2(2m-1)!} \delta^{2m-1}(x) \qquad \ldots(5)$$

on the space $D\{0, 2, \ldots, 2m-2\}$;

$$(x_-^r) \cdot \delta^{r+2m}(x) = -\frac{(r+2m)!}{2(2m)!} \delta^{2m}(x) \qquad \ldots(6)$$

on the space $D\{1, 3, \ldots, 2m-1\}$.

The results (5) and (6) obtained by Fisher (1972b) are not correct.

The object of this paper is to supply a new proof for the products $(x_+^r) \cdot \delta^{r+1}$, $(x_+^r) \cdot \delta^{r+2}$, $(x_+^r) \cdot \delta^{r+2m-1}$, $(x_+^r) \cdot \delta^{r+2m}$ and to correct the results (5) and (6). Our method of obtaining the products $(x_+^r) \cdot \delta^{r+1}, (x_+^r) \cdot \delta^{r+2m}$ is based on the method of obtaining finite part of a divergent integral. By neglecting divergent part of an integral, pseudo functions are defined in (Zemanian 1965).

For pseudo functions and finite part of a divergent integral see Zemanian (1965; pp. 15-18, 57-61).

2. The Pseudo Distribution $(x_+^r) \cdot \delta^{r+2m-1}$

The distribution (x_+^r) is an ordinary summable function defined by

$$x_+^r = \begin{cases} x^r, & x > 0 \\ 0, & x \leqslant 0 \end{cases}$$

for $r = 0, 1, 2, \ldots$.

The function $(x_+^r)_n$ is defined by (see Fisher 1971, p. 294)

$$(x_+^r)_n = (x_+^r) * \delta_n$$

$$= \int_{-1/n}^{x} (x-t)^r \cdot \delta_n(t)\, dt \qquad \ldots(7)$$

The support of $(x_+^r)_n \cdot \delta_n^{r+2m-1}$ is contained in the interval $(-1/n, 1/n)$. We define the distribution $(x_+^r) \cdot \delta^{r+2m-1}$ as

$$\langle (x_+^r) \cdot \delta^{r+2m-1}(x), \phi(x) \rangle$$

$$= \lim_{n \to \infty} \int_{-1/n}^{1/n} (x_+^r)_n \cdot \delta_n^{r+2m-1}(x)\, \phi(x)\, dx. \quad \ldots(8)$$

Substituting the value of $(x_+^r)_n$ from (7) in (8) and changing the order of integration which is permissible here, we have

$$\langle (x_+^r) \cdot \delta^{r+2m-1}(x), \phi(x) \rangle$$

$$= \operatorname{Lim}_{n \to \infty} \int_{-1/n}^{1/n} \delta_n(t) \int_t^{1/n} (x - t)^r\, \delta_n^{r+2m-1}(x)\, \phi(x)\, dx\, dt. \quad \ldots(9)$$

By Taylor's theorem we have

$$\phi(x) = \phi(0) + x\phi'(0) + \frac{x^2}{2!}\phi''(0) + \ldots$$

$$+ \frac{x^{2m-3}}{(2m-3)!}\phi^{2m-3}(0) + \frac{x^{2m-2}}{(2m-2)!}\phi^{2m-2}(0)$$

$$+ \frac{x^{2m-1}}{(2m-1)!}\phi^{2m-1}(0) + \frac{x^{2m}}{(2m)!}\phi^{2m}(hx), \quad 0 \leqslant h \leqslant 1.$$

Substituting value of $\phi(x)$ in (9), we get

$$\langle (x_+^r) \cdot \delta^{r+2m-1}(x), \phi(x) \rangle$$

$$= \int_{-1/n}^{1/n} \delta_n(t) \int_t^{1/n} (x - t)^r\, \delta_n^{r+2m-1}(x) \Big[\phi(0) + \frac{x^2}{2!}\phi''(0) + \ldots$$

$$+ \frac{x^{2m-2}}{(2m-2)!}\phi^{2m-2}(0) \Big]\, dx\, dt$$

$$+ \int_{-1/n}^{1/n} \delta_n(t) \int_t^{1/n} (x - t)^r\, \delta_n^{r+2m-1}(x) \Big[x\phi'(0) + \frac{x^3}{3!}\phi'''(0) + \ldots$$

$$+ \frac{x^{2m-3}}{(2m-3)!}\phi^{2m-3}(0) \Big]\, dx\, dt +$$

(*equation continued on p.* 1085)

$$+ \int_{-1/n}^{1/n} \delta_n(t) \int_t^{1/n} (x-t)^r \, \delta_n^{r+2m-1}(x) \frac{x^{2m-1}}{(2m-1)!} \phi^{2m-1}(0) \, dx \, dt$$

$$+ \int_{-1/n}^{1/n} \delta_n(t) \int_t^{1/n} (x-t)^r \, \delta_n^{r+2m-1}(x) \frac{x^{2m}}{(2m)!} \phi^{2m}(hx) \, dx \, dt$$

$$= \lim_{n \to \infty} [I_1 + I_2 + I_3 + I_4]. \qquad \text{...(10)}$$

We now find values of I_1, I_2, I_3 and I_4 separately.

$$I_1 = \int_{-1/n}^{1/n} \delta_n(t) \int_t^{1/n} (x-t)^r \, \delta_n^{r+2m-1} \Big[\phi(0) + \frac{x^2}{2!}\phi''(0) + \dots$$

$$+ \frac{x^{2m-2}}{(2m-2)!} \phi^{2m-2}(0)\Big] dx \, dt \qquad \text{...(11)}$$

Some computation shows that I_1 is equal to finite sum of terms, a typical term being

$$\int_{-1/n}^{1/n} \delta_n(t) \int_t^{1/n} x^r \, \delta_n^{r+2m-1}(x) \frac{x^{2m-2}}{(2m-2)!} \phi^{2m-2}(0) \, dx \, dt$$

$$= \frac{\phi^{2m-2}(0)}{(2m-2)!} \int_{-1/n}^{1/n} \delta_n(t) \int_t^{1/n} x^{r+2m-2} \, \delta_n^{r+2m-1}(x) \, dx \, dt. \qquad \text{...(12)}$$

Making use of results (see Fisher 1972b, pp. 373-74)

$$\int_t^{1/n} s^p \, \delta_n^r(s) ds = - t^p \, \delta_n^{r-1}(t) + \dots + (-1)^{p+1} p! \, \delta_n^{r-p-1}(t), \text{ for } r > p \qquad \text{...(13)}$$

and

$$\int_t^{1/n} s^r \, \delta_n^r(s) \, ds = - t^r \, \delta_n^{r-1}(t) + \dots$$

$$+ (-1)^r r! \, t \, \delta_n(t) + (-1)^r r! \, [1 - H_n(t)] \qquad \text{...(14)}$$

in (12), we have

$$(12) = \frac{\phi^{2m-2}(0)}{(2m-2)!} \int_{-1/n}^{1/n} \delta_n(t) \, [- t^{r+2m-2} \, \delta_n^{r+2m-2}(t) + \dots] \, dt. \qquad \text{...(15)}$$

A typical term of (15) is

$$- \frac{\phi^{2m-2}(0)}{(2m-2)!} \int_{-1/n}^{1/n} t^{r+2m-2} \, \delta_n(t) \, \delta_n^{r+2m-2}(t) \, dt. \qquad \text{...(16)}$$

Now since $\delta_n(t) = n\rho(nt)$,

$$\delta_n^{r+2m-2}(t) = n^{r+2m-1}\,\rho^{r+2m-2}(nt).$$

Hence

$$(16) = -\frac{\phi^{2m-2}(0)}{(2m-2)!}\int_{-1/n}^{1/n} t^{r+2m-2}\,n\rho(nt)\,n^{r+2m-1}\,\rho^{r+2m-2}(nt).$$

Substituting $nt = x$, we have

$$(16) = n\left[-\frac{\phi^{2m-2}(0)}{(2m-2)!}\int_{-1}^{1}\rho(x)\,x^{r+2m-2}\,\rho^{r+2m-2}(x)\,dx.\right.$$

Similarly finding values of the other terms of (15) we can write

$$I_1 = \{\text{A finite linear combination of positive powers of } n\}\cdot K_1$$

$$= K_1\cdot D_n.$$

Here K_1 is some constant and D_n is symbol to denote a finite linear combination of positive powers of n.

Now consider the integral I_2

$$= \int_{-1/n}^{1/n}\delta_n(t)\int_{t}^{1/n}(x-t)^r\,\delta_n^{r+2m-1}(x)\left[x\phi'(0) + \ldots\right.$$

$$\left. + \frac{x^{2m-3}}{(2m-3)!}\phi^{2m-3}(0)\right]dx\,dt.$$

Some computation shows that I_2 is equal to finite sum of terms, a typical term being

$$\int_{-1/n}^{1/n}\delta_n(t)\int_{t}^{1/n} x^r\,\delta_n^{r+2m-1}(x)\,\frac{x^{2m-3}}{(2m-3)!}\,\phi^{2m-3}(0)\,dx\,dt$$

$$= \frac{\phi^{2m-3}(0)}{(2m-3)!}\int_{-1/n}^{1/n}\delta_n(t)\left[-t^{r+2m-3}\,\delta_n^{r+2m-2}(t) + \ldots\right]dt \quad \ldots(17)$$

by using result (13).

Since $\delta^r(t)$ is even or odd according as r is even or odd (see 1966 p. 88), hence the function

$$\delta_n(t)\,[\,-\,t^{r+2m-3}\,\delta_n^{r+2m-2}(t)+\ldots]$$

is odd, and therefore value of (17) is zero. Other integrals of I_2 can also be obtained similarly. Thus,

$$I_2 = 0.$$

We now consider the integral

$$I_3 = \int_{-1/n}^{1/n} \delta_n(t) \int_t^{1/n} (x-t)^r\, \delta_n^{r+2m-1}(x)\, \frac{x^{2m-1}}{(2m-1)!}\, \phi^{2m-1}(0)\, dx\, dt$$

$$= \frac{\phi^{2m-1}(0)}{(2m-1)!} \int_{-1/n}^{1/n} \delta_n(t) \int_t^{1/n} x^{r+2m-1}\, \delta_n^{r+2m-1}(x)\, dx\, dt$$

all other intregrals in I_3 being zero.

On using the result (14) we have

$$I_3 = \frac{\phi^{2m-1}(0)}{(2m-1)!} \int_{-1/n}^{1/n} \delta_n(t)\,[\,-\,t^{r+2m-1}\,\delta_n^{r+2m-2}(t)$$

$$+ \ldots + (-1)^{r+2m-1}\,(r+2m-1)!\,\{1-H_n(t)\}]\,dt$$

$$= \frac{\phi^{2m-1}(0)}{(2m-1)!} \int_{-1/n}^{1/n} \delta_n(t)\,(-1)^{r+2m-1}\,(r+2m-1)!\,\{1-H_n(t)\}\,dt.$$

Now since

$$\int_{-1/n}^{1/n} \{1-H_n(t)\}\,\delta_n(t) = \tfrac{1}{2}$$

(see Fisher 1971, p. 295)

$$I_3 = \tfrac{1}{2}(-1)^{r+1}\,(r+2m-1)!\,\frac{\phi^{2m-1}(0)}{(2m-1)!}$$

Now consider the integrals I_4.

$$I_4 = \int_{-1/n}^{1/n} \delta_n(t) \int_t^{1/n} (x-t)^r\, \delta_n^{r+2m-1}(x)\, \frac{x^{2m}}{(2m)!}\, \phi^{2m}(hx)\, dx\, dt.$$

Following Fisher (1972b, 1973) we can write

$$I_4 = K_2 \,.\, C_n.$$

Here K_2 is some constant and C_n is a finite linear combination of negative powers of n.

Substituting the values of I_1, I_2, I_3 and I_4 in (10) we have

$$\langle (x_+^r) \,.\, \delta^{r+2m-1}(x), \phi(x) \rangle$$

$$= \lim_{n\to\infty} \left[K_1 D_n + \tfrac{1}{2}(-1)^{r+1}(r+2m-1)! \frac{\phi^{2m-1}(0)}{(2m-1)!} + K_2 C_n \right] \quad ...(18)$$

D_n is a finite linear combination of positive powers of n. As $n \to \infty$, D_n diverges whereas C_n converges to zero. We subtract $K_1 D_n$ from (18). The result is finite part of the divergent integral. We define it as

$$F_p \langle (x_+^r \,.\, \delta^{r+2m-1}(x), \phi(x) \rangle$$

$$= \mathrm{Lim}_{n\to\infty} \left[\tfrac{1}{2}(-1)^{r+1}(r+2m-1)! \frac{\phi^{2m-1}(0)}{(2m-1)!} + K_2 C_n \right]$$

$$= (-1)^{r+1} (\tfrac{1}{2})^{(r+2m-1)!} \frac{\phi^{2m-1}(0)}{(2m-1)!} \quad ...(19)$$

(F_p is notation for finite part).

It is important to note that (19) defines a continuous linear functional (distribution) on D. Linearity of (19) is clear. Moreover if the sequence $\{\phi_v\}_{v=1}^{\infty}$ converges in D to zero, it can be easily proved that the sequence $\{F_p \langle (x_+^r) \,.\, \delta^{r+2m-1}(x), \phi_v \rangle\}_{v=1}^{\infty}$ also converges to zero. This establishes continuity of our functional. Thus (19) defines a distribution. We denote this distribution by $P_f\{(x_+^r) \,.\, \delta^{r+2m-1}\}$ (P_f is notation for pseudo function). Following Schwartz $P_f\{(x_+^r) \,.\, \delta^{r+2m-1}\}$ is a pseudo function. Thus

$$\langle P_f\{(x_+^r) \,.\, \delta^{r+2m-1}\}, \phi(x) \rangle = F_p \langle (x_+^r) \,.\, \delta^{r+2m-1}(x), \phi(x) \rangle$$

$$= (-1)^{r+1} \tfrac{1}{2} (r+2m-1)! \frac{\phi^{2m-1}(0)}{(2m-1)!}$$

$$= (-1)^r \tfrac{1}{2} \frac{(r+2m-1)!}{(2m-1)!} \delta^{2m-1}(x) \quad ...(20)$$

for $r = 0, 1, 2, \ldots$ and $m = 1, 2, \ldots$. This result is similar to the result obtained by Fisher (1972b) for the product $(x_+^r) \,.\, \delta^{r+2m-1}(x)$.

Corollary — Putting $m = 1$ in (20) we have

$$\langle P_f(x_+^r) \,.\, \delta^{r+1}, \phi(x) \rangle$$

$$= (-1)^r \tfrac{1}{2} (r+1)!\, \delta'(x). \qquad \ldots(21)$$

The result (21) is similar to the result obtained for the product $(x_+^r) \,.\, \delta^{r+1}(x)$ by Fisher (1973).

3. The Pseudo Distribution $(x_+^r) \,.\, \delta^{r+2m}(x)$

The support of $(x_+^r)_n \,.\, \delta_n^{r+2m}(x)$ is contained in the interval $(-1/n, 1/n)$. We can write

$$\langle (x_+^r) \,.\, \delta^{r+2m}(x), \phi(x) \rangle = \lim_{n\to\infty} \int_{-1/n}^{1/n} \int_{t}^{1/n} (x-t)^r\, \delta_n^{r+2m}(x)\, \delta_n(t)\, \phi(x)\, dx\, dt. \qquad \ldots(22)$$

The above integral can again be expressed as sum of four integrals I_1, I_2, I_3 and I_4. By finding values of these integrals it can be seen that (22) consists of divergent and convergent parts. Neglecting the divergent part of (22) we define the distribution $P_f\{x_+^r) \,.\, \delta^{r+2m}(x)\}$ as

$$\langle P_f\{(x_+^r) \,.\, \delta^{r+2m}\}, \phi(x) \rangle = F_p \langle (x_+^r) \,.\, \delta^{r+2m}, \phi(x) \rangle$$

$$= \frac{(-1)^r (r+2m)!}{2(2m)!}\, \delta^{2m}(x) \qquad \ldots(23)$$

for $r = 0, 1, 2, \ldots$ and $m = 1, 2, 3, \ldots$

This result is similar to the result obtained by Fisher (1972b) for the product $(x_+^r) \,.\, \delta^{r+2m}(x)$.

Corollary — Substituting $m = 1$ in (23), we get

$$\langle P_f\{(x_+^r) \,.\, \delta^{r+2}\}, \phi(x) \rangle$$

$$= F_p \langle (x_+^r) \,.\, \delta^{r+2}, \phi(x) \rangle$$

$$= \frac{(-1)^r (r+2)!}{2(2)!}\, \delta''(x). \qquad \ldots(24)$$

This is similar to the result obtained by Fisher (1973) for the product $(x_+^r) \,.\, \delta^{r+2}$.

4. The Distributions $P_f\{(x_-^r) . \delta^{r+2m-1}\}$ and $P_f\{(x_-^r) . \delta^{r+2m}\}$

Since

$$x^r . \delta^{r+2m-1}(x) = [(x_+^r) + (-1)^r (x_-^r)]\, \delta^{r+2m-1}(x)$$

$$= \frac{(-1)^r (r+2m-1)!}{(2m-1)!} \delta^{2m-1} \qquad ...(25)$$

and

$$x^r . \delta^{r+2m} = [(x_+^r) + (-1)^r (x_-^r)]\, \delta^{r+2m}(x)$$

$$= \frac{(-1)^r (r+2m)!}{(2m)!} \delta^{2m}(x). \qquad ...(26)$$

The above two results are particular cases of the result

$$t^n \delta^m(t) = \begin{cases} 0 & , \quad m < n \\ (-1)^n n!\, \delta(t) & , \quad m = n \\ (-1)^n \dfrac{m!}{(m-n)!} \delta^{m-n}(t), & m > n \end{cases}$$

given in Zemanian (1965, p. 55, Sec. 2.4, Prob. 10).

Substituting values of $(x_+^r) . \delta^{r+2m-1}(x)$ and $(x_+^r) . \delta^{r+2m}(x)$ from (20) and (23) in (25) and (26) respectively, we get

$$P_f\{(x_-^r) . \delta^{r+2m-1}(x)\} = \frac{(r+2m-1)!}{2(2m-1)!} \delta^{2m-1}(x) \qquad ...(27)$$

and

$$P_f\{(x_-^r) . \delta^{r+2m}(x)\} = \frac{(r+2m)!}{2(2m)!} \delta^{2m}(x). \qquad ...(28)$$

(27) and (28) are the correct values of the distributions $P_f\{(x_-^r) . \delta^{r+2m-1}(x)\}$ and $P_f\{(x_-^r) . \delta^{r+2m}(x)\}$ while the values obtained by Fisher (1972b) for the distributions $(x_-^r) . \delta^{r+2m-1}(x)$ and $(x_-^r) . \delta^{r+2m}$ are not correct.

Corollary — Putting $m = 1$ in (27) and (28), we have

$$Pf\{(x_-^r) . \delta^{r+1}(x)\} = \frac{(r+1)!}{2} \delta'(x) \qquad ...(29)$$

$$Pf\{(x_-^r) . \delta^{r+2}\} = \tfrac{1}{4}(r+2)!\, \delta''(x). \qquad ...(30)$$

The results (29) and (30) are similar to the results obtained by the author (Tiwari 1979).

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