Indian J. pure appl. Math., **10**(9) : 1092-1098, September 1979

VARIATIONAL METHOD TO FORCED CONVECTION HEAT TRANSFER NEAR FORWARD STAGNATION POINT OF A CYLINDER

P. SINGH

Department of Mathematics, Indian Institute of Technology, Kanpur 208016

AND

D. K. BHATTACHARYA

Department of Mathematics, Indian Institute of Technology, Kharagpur 721302

(*Received* 13 *September* 1978)

The governing principle of dissipative processes is applied to study the forced convection heat transfer near the forward stagnation point of a heated cylinder when an incompressible viscous fluid is flowing around it. The rate of heat transfer from the cylinder to the fluid is obtained with the help of variational method based on Gyarmati's principle. It is found that the new approximate results are quite close to the already known theoretical and experimental results.

1. INTRODUCTION

The main aim of this analysis is to apply some recent developments in irreversible thermodynamics to obtain the approximate analytical solution of the forced convection heat transfer near the forward stagnation point of a cylinder. Gyarmati (1969, 1970) formulated a variational principle which describes the evolution of linear, quasi-linear and some non-linear irreversible processes in space and time. The principle, which is called the governing principle of dissipative processes [GPDP] is written as

$$\delta \int_V [\sigma - \Psi - \Phi] \, dV = 0 \qquad ...(1)$$

where integration is taken over the volume V of the system. Here σ denotes the entropy production which is a bilinear expression of f thermodynamic forces $\mathbf{X}_i$ and the conjugated fluxes $\mathbf{J}_i$, i.e.,

$$\sigma = \sum_{i=1}^{f} \mathbf{J}_i \, . \, \mathbf{X}_i \geqslant 0. \qquad ...(2)$$

In the linear Onsager theory the fluxes and forces are related by the following linear constitutive equations

$$\mathbf{J}_i = \sum_{i=1}^{f} L_{ik} \mathbf{X}_k \quad \text{or} \quad X_i = \sum_{k=1}^{f} R_{ik} \mathbf{J}_k \qquad ...(3)$$

where the coefficients L_{ik} and R_{ik} are the conductivities and resistances respectively, the matrices of which are mutually reciprocal and symmetric (see Onsager 1931a, b). The local dissipation potentials Ψ and Φ are defined in the following homogeneous quadratic forms (Gyarmati 1969, 1970)

$$\left.\begin{aligned} \Psi(\mathbf{X}, \mathbf{X}) &\equiv \tfrac{1}{2} \sum_{i,k=1}^{f} L_{ik}\mathbf{X}_i \cdot \mathbf{X}_k \geqslant 0 \\ \Phi(\mathbf{J}, \mathbf{J}) &\equiv \tfrac{1}{2} \sum_{i,k=1}^{f} R_{ik}\mathbf{J}_i \cdot \mathbf{J}_k \geqslant 0 \end{aligned}\right\} \quad \ldots(4)$$

which correspond to the entropy form (2) written in terms of thermodynamic forces and fluxes respectively. The principle (1) may now be written as

$$\delta \int_V \Big[\sum_{i=1}^{f} \mathbf{J}_i \cdot \mathbf{X}_i - \tfrac{1}{2} \sum_{i,k=1}^{f} L_{ik}\mathbf{X}_i \cdot \mathbf{X}_k - \tfrac{1}{2} \sum_{i,k=1}^{f} R_{ik}\mathbf{J}_i \cdot \mathbf{J}_k \Big]\, dV = 0. \quad \ldots(5)$$

It should be noted that the variation of the principle (5) with respect to forces and fluxes simultaneously gives the universal form of the principle. However, the restricted variation of the principle with respect to one of these two variables give two partial forms of it which are known as flux representation and force representation.

Here we introduce a generalized variational method based on GPDP for treating the steady state forced convection heat transfer near the forward stagnation point of a cylinder. In this case the general functional $(\sigma - \Psi - \Phi)$ is obtained from the energy balance. The Euler-Lagrange equation of the functional is the steady state form of the energy equation describing the heat transfer near the stagnation point and therefore the direct approximate solution by variational technique of this functional is the approximate solution of the energy equation itself. It is found that the rate of heat transfer obtained with the help of a third degree temperature profile is quite close to the theoretical and experimental results already known.

2. Formulation and Method of Solution

Consider two-dimensional flow around a cylindrical body with a boundary layer extending from front stagnation point in both the directions around the cylinder. If the curvature of the surface is not very large, then sufficiently near $x = 0$, the velocity u_1 outside the boundary layer increases linearly with the distance so that $u_1 = \beta_1 x$. Here x is measured along the surface from the stagnation point and y is normal to the surface. u and v are velocities along x and y directions inside the boundary layer respectively. If the body is heated to temperature T_w, the energy balance takes the form

$$\rho C_v \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) + \nabla \cdot \mathbf{J}_q = 0 \quad \text{...(6)}$$

where T denotes temperature inside the boundary layer. This equation describes the steady state temperature distribution inside the boundary layer region. The linear constitutive equation in this case has the form

$$J_{q1} = - L_\lambda \frac{\partial T}{\partial y}. \quad \text{...(7)}$$

Here the thermodynamic force is represented in the Fourier picture (Gyarmati 1970) and L_λ is the coefficient of heat conduction and is constant. Considering the entropy production, we get (Gyarmati 1970)

$$T^2 \sigma = - J_{q1} \frac{\partial T}{\partial y} \geqslant 0 \quad \text{...(8)}$$

and the dissipation potentials are

$$T^2 \Psi = \Psi^{**} = \frac{L_\lambda}{2} \left(\frac{\partial T}{\partial y} \right)^2, \; T^2 \Phi = \Phi^{**} = \frac{1}{2L_\lambda} J_{q1}^2 \, .$$

Gyarmati's principle in Fourier picture takes the form

$$\delta \int_V [T^2 \sigma - \Psi^{**} - \Phi^{**}] \, dV = 0,$$

or

$$\delta \int_V \left[- J_{q1} \frac{\partial T}{\partial y} - \frac{L_\lambda}{2} \left(\frac{\partial T}{\partial y} \right)^2 - \frac{1}{2L_\lambda} J_{q1}^2 \right] dV = 0. \quad \text{...(9)}$$

It should be noted now that the principle (9) contains two unknowns J_{q1} and T which are connected with the exact constitutive relation (7). In the dual field method, we assume an approximate constitutive relation between J_{q1} and the temperature T (Singh 1976a, b).

Thus we write the approximate linear relation

$$J_{q1} = - L_\lambda \frac{\partial T^*}{\partial y} \quad \text{...(10)}$$

where T^* is an approximate temperature and satisfies the same conditions as T. In the exact theory $T = T^*$ and the Lagrangian density is zero. Introducing (10) into (6) and (9), we get

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T^*}{\partial y^2} \quad \text{...(11)}$$

$$\delta \int_V \left[\frac{\partial T}{\partial y}\frac{\partial T^*}{\partial y} - \tfrac{1}{2}\left(\frac{\partial T}{\partial y}\right)^2 - \tfrac{1}{2}\left(\frac{\partial T^*}{\partial y}\right)^2\right] dV = 0 \qquad ...(12)$$

where α denotes the thermal diffusivity of the fluid.

Using similarity transformations

$$\left.\begin{aligned} \eta &= \left(\frac{\beta_1}{\nu}\right)^{1/2} y, \qquad u = \beta_1 x f'(\eta) \\ v &= -(\nu\beta_1)^{1/2} f(\eta), \frac{T - T_\infty}{T_w - T_\infty} = \theta(\eta) \end{aligned}\right\} \qquad ...(13)$$

eqn. (11) and the principle (12) yields

$$\frac{d^2\theta^*}{d\eta^2} + \frac{P}{2} f \frac{d\theta}{d\eta} = 0 \qquad ...(14)$$

$$\delta \int_0^L dx \int_0^{d_T} \left[\frac{d\theta^*}{d\eta}\frac{d\theta}{d\eta} - \tfrac{1}{2}\left(\frac{d\theta}{d\eta}\right)^2 - \tfrac{1}{2}\left(\frac{d\theta^*}{d\eta}\right)^2\right] d\eta = 0 \qquad ...(15)$$

where P is the Prandtl number, L is any arbitrary length along x axis and T_∞ denotes the temperature of the free stream. To determine the thermal boundary layer thickness, d_T, we assume following temperature profile of third degree

$$\theta = 1 - \frac{3\eta}{2d_T} + \tfrac{1}{2}\frac{\eta^3}{d_T^3} \qquad ...(16)$$

which satisfies the boundary conditions

$$\left.\begin{aligned} \eta = 0&: \quad \theta = 1, \frac{d^2\theta}{d\eta^2} = 0 \\ \eta = d_T&: \quad \theta = 0, \frac{d\theta}{d\eta} = 0. \end{aligned}\right\} \qquad ...(17)$$

In (16) d_T is the variational parameter which is to be determined with the help of the principle (15). To evaluate the principle, we have to determine θ^* with the help of eqn. (14). To get θ^* we need the velocity profile inside the boundary layer along the cylinder. Under the transformation (13), the momentum equation is (Goldstein 1938)

$$f''' + ff'' - f'^2 + 1 = 0 \qquad ...(18)$$

which is exactly the same as that of two dimensional viscous flow against a flat plate held perpendicular to the oncoming stream. In this case viscous boundary layer is of constant thickness so that the thinning of the layer due to accelerating main stream is just sufficient to balance the thickening due to diffusion of shear. Taking a third degree polynomial for velocity profile inside the boundary layer:

$$f' = \frac{3\eta}{d^2} - \frac{3\eta^2}{d^2} + \frac{\eta^3}{d^3} \quad \text{...(19)}$$

the viscous boundary layer thickness d is obtained with help of GPDP as (Singh and Bhattacharya 1978)

$$d = 2.530. \quad \text{...(20)}$$

Using (16) and (19) in (14), θ^* is obtained as

$$\frac{d\theta^*}{d\eta} = -\frac{3}{2} P\left[\frac{\eta^4}{d_T^3}\left(\frac{1}{28}\frac{\eta^3}{d^3} - \frac{\eta^2}{6d^2} + \frac{3}{10}\frac{\eta}{d}\right) - \frac{\eta^2}{2d_T}\left(\frac{\eta^3}{10d^3} - \frac{\eta^2}{2d^2} + \frac{\eta}{d}\right) + d_T^2\left(\frac{1}{5d} - \frac{d_T}{12d^2} + \frac{1}{70}\frac{d_T^2}{d^3}\right)\right] \quad \text{...(21)}$$

which satisfies the condition $d\theta^*/d\eta = 0$ at the edge of the thermal boundary layer. Substituting (16) and (21) into the principle (15) and integrating with respect to η, we get

$$\delta \int_0^L \left\{ d_T^2 P^2 \left[\frac{d_T^6}{d^6} - 10.012\left(\frac{d_T}{d}\right)^5 + 43.974\left(\frac{d_T}{d}\right)^4 - 93.867\left(\frac{d_T}{d}\right)^3 + 88.021\left(\frac{d_T}{d}\right)^2\right] - 57.08P\left[\left(\frac{d_T}{d}\right)^3 - 4.190\left(\frac{d_T}{d}\right)^2 + 6.250\frac{d_T}{d}\right] - \frac{456.621}{d_T^2}\right\} dx = 0. \quad \text{...(22)}$$

Taking the variation with respect to the parameter d_T, we get following equation as Euler Lagrange's equation of (22)

$$\triangle^{10} - 10.012\,\triangle^9 + 43.974\,\triangle^8 - 93.867\,\triangle^7 + 88.021\,\triangle^6 - \frac{1}{Pd^2} \times [57.08\,\triangle^5 - 239.182\,\triangle^4 + 356.735\,\triangle^3] - \frac{456.621}{P^2d^4} = 0 \quad \text{...(23)}$$

where $\triangle = d_T/d$.

Equation (23) is solved by modified Bairstow method (Hamming 1971). Out of ten roots only one is real and positive and hence corresponds to thermal boundary layer thickness.

3. Results and Discussions

The rate of heat transfer from a section of the cylinder of breadth b and length x measured from stagnation point is

$$Q = \left(\frac{d\theta}{d\eta}\right)_{\eta=0} kbx(T_w - T_\infty)\left(\frac{\beta_1}{\nu}\right)^{1/2}$$

and the corresponding Nusselt number is

$$Nu = \left(\frac{d\theta}{d\eta}\right)_{\eta=0}\left(\frac{\beta_1 l^2}{\nu}\right)^{1/2}$$

where l is a representative length. A set of values of $(d\theta/d\eta)_{\eta=0}$ together with the exact result as reported by Goldstein (1938) is given in Table I for comparison. It is found that the difference between the present values and the already known exact ones is less than 5%. It may be mentioned that the results can be further improved by introducing a higher order polynomial or by increasing the number of variational parameters in trial functions of velocity and temperature profiles.

Table I

P	Approximate value from GPDP	Exact value
0.6	0.442	0.466
0.7	0.471	0.495
0.8	0.498	0.521
0.9	0.522	0.546
1.0	0.544	0.570
1.1	0.565	0.592

Comparison of $(d\theta/d\eta)_{\eta=0}$

The result can be compared with experimental on the temperature distribution in fluid stream. Near the stagnation point of a circular cylinder the theoretical values for the velocity outside boundary layer is approximately equal to

$$u_1 = \frac{4u_0 x}{d_c}$$

where u_0 is the velocity of the mainstream and d_c is the diameter of the cylinder. Hence $\beta_1 = 4u_0/d_c$. Taking P for air as 0.733 and $u_0 d_c/\nu = 4 \times 10^4$, we list the theoretical, approximate and experimental values of Nu (Goldstein 1938):

Values of Nu	Source
202	Exact solution
192	Approximate value from GPDP
206	Experiment by Lohrisch
190	Experiment by Klein
185	Experiment by Drew and Ryan

These results clearly show that the approximate value of Nu obtained with the help of GPDP is in good agreement with theoretical and experimental results. Thus the method is applicable to convective heat transfer boundary layer systems too.

Acknowledgement

The authors are thankful to Prof. I. Gyarmati for many useful suggestions. One of the authors (D. K. B.) is grateful to C.S.I.R., New Delhi, for financial assistance.

References

Goldstein, S. (1938). Modern Developments in Fluid Dynamics, Vol. II. Oxford.

Gyarmati, I. (1969). On the governing principle of dissipative processes and its extension to non-linear problems. *Ann. Phys.*, **23**, 353.

——— (1970). Non-equilibrium Thermodynamics. Field Theory and Variational Principles. Springer Verlag, Berlin.

Hamming, R. W. (1971). Introduction to Applied Numerical Analysis. McGraw-Hill Book Co., Inc., New York.

Onsager, L. (1931a). Reciprocal relations in irreversible processes—I. *Phys. Rev.*, **37**, 405.

——— (1931b). Reciprocal relations in irreversible processes—II. *Phys. Rev.*, **37**, 2265.

Singh, P. (1976a). The application of GPDP to Benerd convection. *Int. J. Heat Mass Transfer*, **19**, 571.

——— (1976b). An approximate technique to thermohydrodynamic stability problem on the basis of GPDP. *J. Non-equilib. Thermodyn.*, **1**, 105.

Singh, P., and Bhattacharya, D. K. (1978). A new approximate method for laminar stagnation flow. *J. Non-equilib. Thermodyn.*, **3**, 103.